SOME PROBABILISTIC ASPECTS OF SET PARTITIONS

Jim Pitman

Technical Report No. 452

Department of Statistics
University of California
367 Evans Hall # 3860
Berkeley, CA 94720-3860

March 11, 1996

1 Introduction

A *partition* of the set $\mathbb{N}_n := \{1, 2, \cdots, n\}$ is an unordered collection of non-empty subsets of $\mathbb{N}_n$. Let $\mathbb{P}_n$ denote the set of all such partitions, and let $B_n = \#(\mathbb{P}_n)$, the number of partitions of $\mathbb{N}_n$. The numbers B_n are known as the *Bell numbers* [4, 3]. See Rota [35] for a survey of their properties and applications. Dobinski [13] discovered the remarkable formula

$$B_n = e^{-1} \sum_{m=1}^{\infty} \frac{m^n}{m!} \qquad (n = 1, 2, \cdots) \tag{1}$$

which leads ([26] 1.9) to the asymptotic evaluation

$$B_n \sim \frac{1}{\sqrt{n}} \lambda(n)^{n+1/2} e^{\lambda(n)-n-1} \qquad \text{as } n \to \infty \tag{2}$$

where $\lambda(n)\log(\lambda(n)) = n$. As noted by Comtet [9], for each n the infinite sum in (1) can be evaluated as the least integer greater than the sum of the first $2n$ terms.

From a probabilistic perspective, the right side of Dobinski's formula represents the nth moment of the Poisson distribution with mean 1. So the initially surprising fact that this expression yields an integer for all n amounts to the fact that all positive integer moments of the Poisson(1) distribution are integers. As explained in Section 2, Dobinski's formula follows easily from the fact that the factorial moments of the Poisson(1) distribution are identically equal to 1, an identity which can be understood probabilistically with essentially no calculation.

While such probabilistic interpretations of various identities related to set partitions are the main theme of this paper, section (1.2) offers an elementary combinatorial proof of Dobinski's formula which seems simpler than other proofs in the literature (Rota [35], Berge [5], p. 44, Comtet [9], p. 211). This argument involves identities whose probabilistic interpretations are brought out later in the paper.

1.1 Notation

Following the notation of [20], let $\left\{ {n \atop k} \right\}$ denote the number of partitions of $\mathbb{N}_n$ into exactly k distinct non-empty subsets, so that

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \tag{3}$$

The $\left\{ {n \atop k} \right\}$ are known as the *Stirling numbers of the second kind*. Let $m^{\underline{k}}$ denote the falling factorial with k factors

$$m^{\underline{k}} = m(m-1)\cdots(m-k+1) \tag{4}$$

which, for positive integers m and k, is the number of permutations of length k of m distinct symbols. The formula

$$m^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} m^{\underline{k}} \tag{5}$$

decomposes the number m^n of sequences of m distinct symbols of length n as the sum over k of the number of such sequences that contain exactly k distinct symbols ([38], p 35.). As an identity of polynomials in m of degree n this identity provides an alternative definition of the coefficients $\left\{ {n \atop k} \right\}$ for $1 \le k \le n$. See [9, 33, 34, 38]. for background and a wealth of further information about Stirling numbers.

1.2 A quick proof of Dobinski's formula.

Divide (5) by $m!$ to obtain for positive integer m and n

$$\frac{m^n}{m!} = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{1}{(m-k)!} \tag{6}$$

This is the identity of coefficients of λ^m in the power series identity

$$\sum_{m=1}^{\infty} \frac{m^n}{m!} \lambda^m = \left(\sum_{k=1}^{n} \left\{ {n \atop k} \right\} \lambda^k \right) \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \right) \tag{7}$$

which rearranges as to give the following the following *horizontal generating function* for the Stirling numbers of the second kind:

$$\sum_{k=1}^{n} \left\{ {n \atop k} \right\} \lambda^k = e^{-\lambda} \sum_{m=1}^{\infty} \frac{m^n}{m!} \lambda^m \tag{8}$$

Take $\lambda = 1$ and use (3) to obtain Dobinski's formula (1).

2 Moments

For a non-negative integer valued random variable X with

$$P(X = m) = p_m \qquad (m = 0, 1, \cdots) \tag{9}$$

and a non-negative function f let

$$E[f(X)] := \sum_m p_m f(m) \tag{10}$$

called the *expected value of* $f(X)$ for X with distribution (9). See [16, 31] for background. From (5) and the linearity of the expectation operator E there is the following well known formula for $E[X^n]$, the nth *moment of* X, in terms of $E[X^{\underline{k}}]$, the kth *factorial moment* of X for $1 \le k \le n$ [33, 10]:

$$E[X^n] = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} E[X^{\underline{k}}] \tag{11}$$

For $\lambda > 0$ let X_λ denote a random variable with the Poisson distribution

$$P(X_\lambda = m) = e^{-\lambda} \frac{\lambda^m}{m!} \qquad (m = 0, 1, \cdots) \tag{12}$$

so that

$$E[f(X_\lambda)] = e^{-\lambda} \sum_{m=0}^{\infty} f(m) \frac{\lambda^m}{m!} \tag{13}$$

Take $f(m) = m^n$ to see that the right side of (8) equals $E[X_\lambda^n]$. So the identity (8) amounts to the formula

$$E[X_\lambda^n] = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \lambda^k \qquad (n = 1, 2, \cdots) \tag{14}$$

for the moments of the Poisson(λ) distribution [32, 30]. This formula is the particular case of (11) for X with Poisson(λ) distribution, for it is known [32, 10] that

$$E[X_\lambda^{\underline{k}}] = \lambda^k \qquad (k = 1, 2, \cdots) \tag{15}$$

Formula (15) follows easily from (13) with $f(m) = m^{\underline{k}}$ by change of summation variable from m to $j = m - k$. In particular, for $\lambda = 1$ the factorial moments of the Poisson(1) distribution are identically equal to 1. So Dobinski's formula (1) can be read from (14) for $\lambda = 1$, which follows as indicated above from from (11) and (15). In essence, this is Rota's [35] proof of Dobinski's formula cast in probabilistic notation. This argument differs from the proof in Section 1.2 in that it involves checking (15) for $\lambda = 1$.

Formula (15) has the following interpretation in terms of a *Poisson process* [24, 31]. Let

$$0 < U_{(1)} < \cdots < U_{(X_\lambda)} < 1 \tag{16}$$

denote the random locations in $(0, 1)$ of the points of a homogeneous Poisson process on $(0, 1)$ with mean intensity measure $\lambda\, du$ for $0 < u < 1$. For each

$k = 1, 2, \cdots$ define an associated *k-tuple point process*, with points in $(0,1)^k$, to have a point at each of the locations $(U_{(\sigma_1)}, \cdots, U_{(\sigma_k)})$ as σ ranges over the $X_\lambda^{\underline{k}}$ different permutations of $\{1, \cdots, X_\lambda\}$ of length k. For distinct $u_i \in (0,1)$, independence properties of the basic Poisson process on $(0,1)$ imply that the mean intensity of the k-tuple point process at $(u_1, \cdots, u_k) \in (0,1)^k$ is

$$\frac{P(\text{some } U_{(\sigma_i)} \in du_i \text{ for each } 1 \leq i \leq k)}{du_1 \cdots du_k} = \frac{(\lambda\, du_1) \cdots (\lambda\, du_k)}{du_1 \cdots du_k} = \lambda^k \tag{17}$$

So the expected number of points in the k-tuple point process is

$$E[X_\lambda^{\underline{k}}] = \lambda^k \int_0^1 du_1 \cdots \int_0^1 du_k = \lambda^k \tag{18}$$

See [33, 8] for various applications of Stirling numbers and their generalizations to the computation of moments of probability distributions. Moments of the normal distribution also have interesting combinatorial interpretations [14, 19]. More generally, the idea of representing combinatorially defined numbers by an infinite sum or an integral, typically with a probabilistic interpretation, has proved to be a very fruitful one. Other examples are the representation of $n!$ as a gamma integral, which leads to Stirling's formula [7, 12, 27], and Laplace's representation of kth differences of powers [25, 10, 23], which yields an asymptotic formula for the Stirling numbers of the second kind. See [29] for a recent survey of asymptotic enumeration methods.

3 Variations of Dobinski's Formula

The derivation of Dobinski's formula given the previous section yields the following proposition:

Proposition 1 *Let X be a random variable with probability distribution (p_m) on $\{0, 1, 2, \cdots\}$ and let n be a positive integer. The following conditions are equivalent:*

(i) the first n factorial moments of X are identically equal to 1;

(ii) the kth moment of X equals B_k for every $1 \leq k \leq n$.

It is well known that for each $\lambda > 0$ the Poisson(λ) distribution is uniquely determined by its moments. See for instance [6], Section 30. So the Poisson(1) distribution is the unique probability distribution whose nth moment is B_n for every n. But for each fixed n there are many probability distributions on $\{0, 1, 2, \cdots\}$ which have the same first n moments as Poisson(1). It is obvious that there can be at most one such distribution of X with $P(X \leq n) = 1$, because the moment conditions amount to a system of n linearly independent equations in n unknowns $p_1, \cdots, p_n$. Less obvious is the fact that the unique solution of these equations is such that $p_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} p_i \leq 1$, so that $(p_1, \cdots, p_n)$ is the restriction to $\{1, \cdots, n\}$ of a unique probability distribution on $\{0, 1, \cdots, n\}$. But this probability distribution on $\{0, 1, 2, \cdots, n\}$ whose first n factorial moments are identically equal to one, is known to arise in the setting of the classical matching problem [10, 16]. If M_n is the number of fixed points of a uniformly distributed random permutation of $\mathbb{N}_n$, then it is easily shown by the method of indicators that the first n factorial moments of M_n are identically equal to 1. See [10]. The distribution of a random variable X with range $\{0, 1, \cdots, n\}$ is recovered from its factorial moments by the classical sieve formula [10]:

$$P(X = m) = \frac{1}{m!} \sum_{k=m}^{n} \frac{(-1)^{m-k} E[X^{\underline{k}}]}{(m-k)!} \qquad (m = 0, 1, \cdots n) \tag{19}$$

For $X = M_n$ with $E[M_n^{\underline{k}}] \equiv 1$ for $0 \leq k \leq n$, this simplifies to

$$P(M_n = m) = \frac{1}{m!} \sum_{s=0}^{n-m} \frac{(-1)^s}{s!} \qquad (m = 0, 1, \cdots n) \tag{20}$$

See Section IV.4 of [15] for further discussion. According to Proposition (1), the kth moment of M_n is B_k for every $1 \leq k \leq n$. That is to say

$$B_k = \sum_{m=1}^{n} \frac{m^k}{m!} \sum_{s=0}^{n-m} \frac{(-1)^s}{s!} \qquad (1 \leq k \leq n) \tag{21}$$

This variation of Dobinkski's formula is derived in quite a different way by Wilf [42] p.22 by substitution of the classical formula

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \tag{22}$$

into (3). As observed by Wilf, Dobinski's formula (1) follows easily from (21) by letting $n \to \infty$. See also Lovász [26],1.9, for a similar argument.

3.1 The moment generating function.

Consider now the *moment generating function (m.g.f.)* of the Poisson(λ) distribution:

$$E[\exp(\theta X_\lambda)] = E\left[\sum_{n=0}^{\infty} \frac{\theta^n X_\lambda^n}{n!}\right] = \sum_{n=0}^{\infty} E[X_\lambda^n] \frac{\theta^n}{n!} \tag{23}$$

where the series converge for all real θ and the switch of E and $\sum$ is easily justified. See [6] for a modern treatment of m.g.f's. From (13) with $f(m) = e^{\theta m}$ there is the standard formula

$$E[\exp(\theta X_\lambda)] = e^{-\lambda} \sum_{m=0}^{\infty} \frac{(\lambda e^\theta)^m}{m!} = \exp(\lambda(e^\theta - 1)) \tag{24}$$

This combines with (8) to yield the following *double generating function* of the Stirling numbers of the second kind. This classical formula (see e.g. Comtet [9], p 206) is an identity between two different expressions for the m.g.f. in θ of the Poisson (λ) distribution:

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{\lambda^k \theta^n}{n!} = \exp(\lambda(e^\theta - 1)) \tag{25}$$

In particular, for $\lambda = 1$ this reduces by (3) to Bell's [4, 3] formula

$$1 + \sum_{n=1}^{\infty} B_n \frac{\theta^n}{n!} = \exp(e^\theta - 1) \tag{26}$$

which gives two expressions for the m.g.f. of the Poisson(1) distribution. Equating coefficients of λ^k in (25) yields the *vertical generating function* of the Stirling numbers of the second kind:

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{\theta^n}{n!} = \frac{1}{k!}(e^\theta - 1)^k \qquad (k = 1, 2, \cdots) \tag{27}$$

See [9, 33, 38] for alternative derivations of these identities. There are similar identities for many other arrays of combinatorial numbers, such as the binomial coefficients and Stirling numbers of the first kind (see e.g. [9, 42], [20],p. 351), most of which admit probabilistic interpretations. Formulae with binomial coefficients typically involve independent trials, while those with Stirling numbers of the first kind typically involve the cycle structure of random permutations [1]. See also [2] for probabilistic analysis of more general combinatorial structures and further references.

4 Random Partitions

A *random partition of* $\mathbb{N}_n$ is a random variable Π with values in the set $\mathbb{P}_n$ of partitions of $\mathbb{N}_n$. The *distribution of* Π then refers to the collection of probabilities $P(\Pi = \pi)$ as π ranges over $\mathbb{P}_n$. Questions about enumeration of partitions of $\mathbb{N}_n$ of various kinds can be phrased probabilistically in terms of a *uniform random partition*, that is a random partition Π with the uniform distribution $P(\Pi = \pi) = 1/B_n$ for each partition $\pi \in \mathbb{P}_n$. For developments of this idea see [22, 21, 36, 18]. Random partitions with non-uniform distribution also arise naturally in various contexts. So it is useful to have models for random partitions, both uniform and non-uniform.

The following *random allocation scheme* is the simplest way to generate a random partition of $\mathbb{N}_n$. See [10, 40, 41] for extensive study of this and related schemes, and further references. Throw n balls labelled by $\mathbb{N}_n$ into m boxes labelled by $\mathbb{N}_m$, and assume all m^n possible allocations of balls into boxes are equally likely. Let Π_{nm} be the partition of balls by boxes. More formally, let X_i be the number of the box containing the ith ball for $1 \leq i \leq n$. Then the X_i are independent and uniformly distributed on $\mathbb{N}_m$, and Π_{nm} is the partition of $\mathbb{N}_n$ induced by the random equivalence relation $i \sim j$ iff $X_i = X_j$. Formally, the X_i can be regarded as co-ordinate maps defined on $(\mathbb{N}_n)^m$, and Π_{nm} is then defined as a map from $(\mathbb{N}_n)^m$ to $\mathbb{P}_n$, the set of partitions of $\mathbb{N}_n$. Let $\#(\pi)$ denote the number of subsets in a partition $\pi \in \mathbb{P}_n$. The distribution of Π_{nm} induced by the uniform distribution P on $\mathbb{N}_m$ can be read from formula (5):

$$P(\Pi_{nm} = \pi) = \frac{m^{\underline{k}}}{m^n} \qquad \text{if } \#(\pi) = k \tag{28}$$

The distribution of $\#(\Pi_{nm})$, the number of occupied boxes when n balls are thrown into m boxes, is given by the following probabilistic equivalent of (5):

$$P[\#(\Pi_{nm}) = k] = \left\{ {n \atop k} \right\} \frac{m^{\underline{k}}}{m^n} \qquad (1 \leq k \leq n) \tag{29}$$

Because the probability displayed in (28) depends on the number of occupied boxes k, this random partition Π of $\mathbb{N}_n$ has a non-uniform distribution for all $n, m \geq 2$. However, as observed by Stam [37], for each fixed n it is possible to generate a uniformly distributed random partition Π of $\mathbb{N}_n$ by a suitable randomization of m. The following Proposition was suggested by Stam's construction, which is described in Corollary 3.

Proposition 2 *Let Π_{nM} denote the random partition of $\mathbb{N}_n$ generated by random allocation of n balls into M boxes, where M is random and given $M = m$ the n balls are thrown independently and uniformly at random into m boxes. Then the following conditions are equivalent:*

(i) Π_{nM} has uniform distribution over all partitions of $\mathbb{N}_n$;

(ii) the distribution of M is of the form

$$P(M = m) = \frac{m^n p_m}{B_n} \qquad (m = 1, 2, \cdots) \tag{30}$$

for some probability distribution (p_m) on $\{0, 1, 2, \cdots\}$ whose first n factorial moments are identically equal to 1.

Before the proof, here are two corollaries that follow immediately from the Proposition and the discussion of Sections 2 and 3:

Corollary 3 [37] *If M has the distribution (30) for $p_m = e^{-1}/m!$, then Π_{nM} is uniform.*

Corollary 4 *The unique distribution of M such $P(M \le n) = 1$ and Π_{nM} is uniform is defined by (30). for $p_m = P(M_n = m)$ as in (30), with M_n the number of fixed points of a random permutation of $\mathbb{N}_n$.*

Proof of Proposition (2). By conditioning on M and using (28),

$$P(\Pi_{nM} = \pi) = \sum_{m=1}^{\infty} \frac{m^{\underline{k}}}{m^n} P(M = m) \text{ if } \#(\pi) = k \tag{31}$$

So the distribution of Π is uniform on $\mathbb{P}_n$ iff

$$\sum_{m=1}^{\infty} \frac{m^{\underline{k}}}{m^n} P(M = m) = \frac{1}{B_n} \qquad (1 \le k \le n) \tag{32}$$

Define

$$p_m = B_n m^{-n} P(M = m) \quad (m = 1, 2, \cdots) \tag{33}$$

so that (32) becomes

$$\sum_{m=1}^{\infty} m^{\underline{k}} p_m = 1 \qquad (1 \leq k \leq n) \tag{34}$$

which for $k = 1$ implies that $\sum_{m=1}^{\infty} p_m \leq \sum_{m=1}^{\infty} m p_m = 1$. So Π_{nM} is uniform iff (p_m) derived from the distribution of M via (33) is the restriction to $\{1, 2, \cdots\}$ of a probability distribution on $\{0, 1, 2, \cdots\}$ whose first n factorial moments are equal to 1. This is condition (ii). □

As shown by Stam, Corollary 3 allows numerous results regarding the asymptotic distribution for large n of a uniform random partition of $\mathbb{N}_n$ to be deduced from corresponding results for the classical occupancy problem defined by random allocations of balls in boxes, for which see [40, 41]. See also [22, 21, 36, 11, 18, 2] for a more detailed account of the asymptotics of uniform random partitions of $\mathbb{N}_n$.

As a variation, the following Corollary is easily obtained by a similar argument:

Corollary 5 *Suppose that M has the distribution*

$$P(M = m) = \frac{m^n P(X_\lambda = m)}{\mu_n(\lambda)} \qquad (m = 1, 2, \cdots) \tag{35}$$

where X_λ has the Poisson(λ) distribution (12), and $\mu_n(\lambda) = E(X_\lambda^n)$. Then the distribution of Π_{nM} is given by

$$P(\Pi_{nM} = \pi) = \frac{\lambda^k}{\mu_n(\lambda)} \qquad \textit{if } \#(\pi) = k \tag{36}$$

As a check, (36) implies

$$P[\#(\Pi_{nM}) = k) = \left\{ {n \atop k} \right\} \frac{\lambda^k}{\mu_n(\lambda)} \qquad (1 \leq k \leq n) \tag{37}$$

so the fact that these probabilities sum to 1 amounts to formula (14) for $\mu_n(\lambda)$. The distribution of Π_{nM} defined by formula (36) defines a *Gibbs' measure* on partitions of $\mathbb{N}_n$. See Steele [39] for further discussion of such measures on combinatorial objects. See also Nijenhuis and Wilf [28] for a recursive algorithm for construction of a uniform random partition of $\mathbb{N}_n$ based on the recurrence

$$B_n = 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} B_k \tag{38}$$

where the right side counts the number of partitions π of $\mathbb{N}_n$ acording to the size k of the subset in π that contains n. ([26] Problem 1.10). For a recent systematic approach to the random generation of labelled combinatorial structures, and further references on this topic, see [17].

References

[1] R. Arratia and S. Tavaré. The cycle structure of random permutations. *Ann. Prob.*, 20:1567–1591, 1992.

[2] R. Arratia and S. Tavaré. Independent process approximations for random combinatorial structures. *Adv. Math.*, 1994.

[3] E. T. Bell. Exponential numbers. *Am. Math. Monthly*, 41:411–419, 1934.

[4] E. T. Bell. Exponential polynomials. *Annals of Mathematics*, 35:258–277, 1934.

[5] C. Berge. *Principles of Combinatorics.* Academic Press, New York and London, 1971.

[6] P. Billingsley. *Probability and Measure.* Wiley, 1995. 2nd ed.

[7] C.R. Blyth and P.K. Pathak. A note on easy proofs of Stirling's theorem. *Amer. Math. Monthly*, 93:376–379, 1986.

[8] Ch. A. Charalambides and Jagbir Singh. A review of the Stirling numbers, their generalizations and statistical applications. *Commun. Statist.-Theory Meth.*, 17:2533–2595, 1988.

[9] L. Comtet. *Advanced Combinatorics.* D. Reidel Pub. Co., 1974. (translated from French).

[10] F.N. David and D.E. Barton. *Combinatorial Chance.* Griffins, London, 1962.

[11] J. M. DeLaurentis and B. G. Pittel. Counting subsets of the random partition and the 'Brownian bridge' process. *Stochastic Processes and their Applications*, 15:155 – 167, 1983.

[12] P. Diaconis and D. Freedman. An elementary proof of Stirling's formula. *Amer. Math. Monthly*, 93:123–125, 1986.

[13] Dobiński. Summirung der Reihe $\sum n^m/n!$ für $m = 1, 2, 3, 4, 5, \ldots$. *Grunert Archiv (Arch. für Mat. und Physik)*, 61:333–336, 1877.

[14] E. B. Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes. *J. Funct. Anal.*, 55:344 – 376, 1984.

[15] W. Feller. *An Introduction to Probability Theory and its Applications, Vol 2.* Wiley, 1966.

[16] W. Feller. *An Introduction to Probability Theory and its Applications, Vol 1,3rd ed.* Wiley, 1968.

[17] Ph. Flajolet, P. Zimmerman, and B.V. Cutsem. A calculus for the random generation of labelled combinatorial structures. *Theoretical Computer Science*, 132:1–35, 1994.

[18] B. Fristedt. The structure of partitions of large sets. Technical Report 86-154, University of Minnesota Mathematics Dept., 1987.

[19] C.D. Godsil. *Algebraic Combinatorics.* Chapman & Hall, New York, 1993.

[20] R. L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics: a foundation for computer science. 2nd ed.* Addison-Wesley, Reading, Mass., 1989.

[21] J. Haigh. Random equivalence relations. *Journal of Combinatorial Theory*, 13:287–295, 1972.

[22] L. H. Harper. Stirling behavior is asymptotically normal. *Ann. Math. Stat.*, 38:410–414, 1966.

[23] L. Holst. On numbers related to partitions of unlike objects and occupancy problems. *Europ. J. Combinatorics*, 2:231–237, 1981.

[24] J.F.C. Kingman. *Poisson Processes*. Clarendon Press, Oxford, 1993.

[25] P.S. Laplace. *Théorie Analytique des Probabilités (2nd Ed.)*. Paris, 1814.

[26] L. Lovász. *Combinatorial Problems and Exercises, 2nd ed.* North-Holland, Amsterdam, 1993.

[27] V. Namias. A simple derivation of Stirling's asymptotic series. *Amer. Math. Monthly*, 93:25–29, 1986.

[28] A. Nijenhuis and H.S. Wilf. A method and two algorithms on the theory of partitions. *J. Combin. Theory A*, 18:219–222, 1975.

[29] A.M. Odlyzko. Asymptotic enumeration methods. In R.L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics Vol. II*, pages 1063–1229. Elsevier, 1995.

[30] C. Philipson. A note on the moments of a Poisson probability distribution. *Skand. Aktuarietidskrift*, 46:243–4, 1963.

[31] J. Pitman. *Probability*. Springer-Verlag, 1993.

[32] J. Riordan. Moment recurrence relations for the binomial Poisson and hypergeometric probability distributions. *Ann. Math. Stat.*, 8:103–111, 1937.

[33] J. Riordan. *An Introduction to Combinatorial Analysis*. Wiley, 1958.

[34] J. Riordan. *Combinatorial Identities*. Wiley, New York, 1968.

[35] G.C. Rota. The number of partitions of a set. *Amer. Math. Monthly*, 71:498–504, 1964.

[36] V. N. Sachkov. Random partitions of sets. *Theory Probab. Appl.*, 19:184–190, 1973.

[37] A.J. Stam. Generation of a random partition of a set by an urn model. *J. Combin. Theory A*, 35:231–240, 1983.

[38] R. P. Stanley. *Enumerative Combinatorics, Vol I.* Wadsworth & Brooks/Cole, 1986.

[39] J. M. Steele. Gibbs' measures on combinatorial objects and the central limit theorem for an exponential family of random trees. *Probability in the Engineering and Informational Sciences*, 1:47–59, 1987.

[40] V. F. Kolchin and B.A. Sevastyanov and V.P. Christyakov. *Random Allocations.* V.H. Winston & sons, Washington, D. C., 1978.

[41] V. A. Vatutin and V. G. Mikhailov. Limit theorems for the number of empty cells in an equiprobable scheme for group allocation of particles. *Theory of Probability and its Applications (Transl. of Teorija Verojatnostei i ee Primenenija)*, 27:734 – 743, 1982.

[42] H. Wilf. *Generatingfunctionology.* Academic Press, San Diego, 1990.