Algebraic evaluations of some Euler integrals, duplication formulae for Appell's hypergeometric function F_1, and Brownian variations *

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Abstract

Explicit evaluations of the symmetric Euler integral $\int_0^1 u^\alpha(1-u)^\alpha f(u)du$ are obtained for some particular functions f. These evaluations are related to duplication formulae for Appell's hypergeometric function F_1 which give reductions of $F_1(\alpha, \beta, \beta, 2\alpha, y, z)$ in terms of more elementary functions for arbitrary β with $z = y/(y-1)$ and for $\beta = \alpha + \frac{1}{2}$ with arbitrary y, z. These duplication formulae generalize the evaluations of some symmetric Euler integrals implied by the following result: if a standard Brownian bridge is sampled at time 0, time 1, and at n independent random times with uniform distribution on $[0, 1]$, then the broken line approximation to the bridge obtained from these $n+2$ values has a total variation whose mean square is $n(n+1)/(2n+1)$.

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1 Introduction

This paper is concerned with the explicit evaluation of some integrals of Euler type

$$\int_0^1 u^{a-1}(1-u)^{b-1} f(u)\, du \tag{1}$$

for particular functions f, especially in the symmetric case $a = b$. These evaluations are related to various reduction formulae for hypergeometric functions represented by such integrals. Sections 4 and 5 explain how we were led to consider such integrals by a simple formula found in [24], for the mean square of the total variation of a discrete approximation to a Brownian bridge obtained by sampling the bridge at n independent random times with uniform distribution on $(0,1)$.

We first recall the basic Euler integrals which define the beta function

$$B(a,b) := \int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \tag{2}$$

for a and b with positive real parts, and Gauss's hypergeometric function

$$F\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| z\right) = \frac{1}{B(b, c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\, dt = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{3}$$

where in the integral it is assumed that $z \notin [1, \infty)$ and that the real parts of b and $c - b$ are positive, and for the series $|z| < 1$ and

$$(\theta)_n := \prod_{i=0}^{n-1} (\theta + i) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)}.$$

We note for ease of later reference the well known consequence of the series expansion in (3) that for any b

$$F\left(\left.\begin{matrix} a, b \\ b \end{matrix}\right| z\right) = (1-z)^{-a}, \tag{4}$$

the Euler transformation [3]

$$F\left(\left.\begin{matrix} a, c+d \\ c \end{matrix}\right| z\right) = (1-z)^{-a-d} F\left(\left.\begin{matrix} -d, c-a \\ c \end{matrix}\right| z\right), \tag{5}$$

and Pfaff-Kummer transformation

$$F\left(\left.\begin{matrix} a, c+d \\ c \end{matrix}\right| z\right) = (1-z)^{-a} F\left(\left.\begin{matrix} a, -d \\ c \end{matrix}\right| \frac{z}{z-1}\right). \tag{6}$$

By expansion of $(1-yu)^{-p}$ and $(1-zu)^{-q}$ in powers of yu and zu, for all positive real a, b, p, q there is the classical evaluation [23]

$$\int_0^1 \frac{u^{a-1}(1-u)^{b-1}\, du}{(1-yu)^p (1-zu)^q} = B(a,b) F_1(a, p, q, a+b; y, z) \tag{7}$$

where F_1 is one of Appell's hypergeometric functions of two variables [4], whose series expansion is

$$F_1(\alpha,\beta,\beta',\gamma;y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\, m!\, n!}\, y^m z^n \qquad (|y|<1, |z|<1). \tag{8}$$

Appell [4, §4] gave a number of reduction formulae which allow F_1 to be expressed in terms of simpler functions when its arguments are subject to various constraints. In particular, Appell gave formulae for $F_1(\alpha,\beta,\beta',\gamma;y,ty)$, and $F_1(\alpha,\beta,\beta',\beta+\beta';y,z)$ in terms of Gauss's hypergeometric function. These reduction formulae for F_1, and various transformations between F_1 and Appell's three other hypergeometric functions of two variables y and z, commonly denoted F_2, F_3 and F_4, can be found in many places [5, 6, 8, 10, 28].

Some of the main results of this paper are the formulae (9) and (11) stated below, as well as their generalization stated throughout the paper. These *duplication formulae* for the Appell function F_1 are reductions in the special case when $\gamma = 2\alpha$ and $\beta' = \beta$, which do not seem to appear in the classical sources. For all complex α, β, y with $|y| < 1/2$

$$F_1\left(\alpha,\beta,\beta,2\alpha;y,\frac{y}{y-1}\right) = F\left(\left.\begin{matrix}\alpha,\beta\\ \alpha+\frac{1}{2}\end{matrix}\,\right|\,\frac{y^2}{4(y-1)}\right). \tag{9}$$

Section 2 derives this formula, explains its close relation to the well known duplication formulae for the sine and gamma functions, and gives a generalization to one of Lauricella's multiple hypergeometric functions. In view of (4), the case $\beta = \alpha + \frac{1}{2}$ of (9) reduces to

$$F_1\left(\alpha,\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2},2\alpha;y,\frac{y}{y-1}\right) = \left(1-\frac{y^2}{4(y-1)}\right)^{-\alpha} = \frac{(1-y)^{\alpha}}{(1-\frac{1}{2}y)^{2\alpha}}. \tag{10}$$

Section 6 offers an interpretation of this formula in terms of a family of probability densities on $(0,1)$ parameterized by $y \in [0,1)$. Formulae (9) and (5) show that

$$F_1\left(\alpha,\alpha+\tfrac{1}{2}+d,\alpha+\tfrac{1}{2}+d,2\alpha;y,\frac{y}{y-1}\right)$$

is an algebraic function of y for each α and each positive integer d. Formula (10) is the particular case $z = y/(y-1)$ of another duplication formula for F_1 which we learned from Ira Gessel: for all complex α, y, z with $|y| < 1$ and $|z| < 1$

$$F_1(\alpha,\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2},2\alpha;y,z) = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-y}\sqrt{1-z}}\right)\left(\frac{2}{\sqrt{1-y}+\sqrt{1-z}}\right)^{2\alpha}. \tag{11}$$

Gessel's proof of (11) is presented in Section 3 together with some generalizations of (11). By (7), for $\alpha > 0$ the right side of (11) gives an algebraic evaluation of

$$\frac{1}{B(\alpha,\alpha)}\int_0^1 \frac{[u(1-u)]^{\alpha-1}\,du}{[(1-yu)(1-zu)]^{\alpha+1/2}} \tag{12}$$

and hence of other Euler integrals by differentiating with respect to y and z. See also [7, 8], [16, (15)] and [31, 30] regarding various other reduction formulae for hypergeometric functions which involve duplication of an argument.

In the process of trying to understand (9)-(11) we realized that our approach gives more general results. Examples are (26) and (28). The referee pointed out a different way to prove (9)-(11). His proof is in Section 8.

2 Duplication formulae and symmetric Euler Integrals

Consider first the elementary duplication formula for the square of the sine function:

$$\sin^2 2\Theta = 4\sin^2\Theta(1-\sin^2\Theta). \tag{13}$$

If Θ is picked uniformly at random from $[0, 2\pi]$, then so is 2Θ modulo 2π, and hence $\sin^2 2\Theta \stackrel{d}{=} \sin^2\Theta$ where $\stackrel{d}{=}$ denotes equality in distribution of two random variables. As shown by Lévy[20] (see also [12, 13, 25]) various random variables A with

$$A \stackrel{d}{=} \sin^2\Theta \tag{14}$$

arise naturally in the study of a one-dimensional Brownian motion B. One such A is the first time that B attains its minimum value on $[0,1]$; another is the amount of time that B spends positive during the time interval $[0,1]$. For A as in (14) the duplication formula (13) is reflected by the identity in distribution

$$4A(1-A) \stackrel{d}{=} A. \tag{15}$$

In other words, the *arcsine distribution* of A on $(0,1)$, with density

$$P(A \in du)/du = \pi^{-1}u^{-1/2}(1-u)^{-1/2} \qquad\qquad (0 < u < 1)$$

is invariant under the transformation $u \to 4u(1-u)$. In the theory of iterated maps [21, Example 1.3] this observation is usually attributed to Von Neumann and Ulam [32]. In purely analytic terms, the identity (15) states that for all non-negative measurable functions h

$$\frac{1}{\pi}\int_0^1 \frac{h(4u(1-u))du}{u^{1/2}(1-u)^{1/2}} = \frac{1}{\pi}\int_0^1 \frac{h(t)dt}{t^{1/2}(1-t)^{1/2}} \tag{16}$$

The substitution $g(x) = \pi^{-1}h(4x)/\sqrt{x}$ reduces (16) to the following expression of the change of variable $t = 4u(1-u)$: for every non-negative measurable g:

$$\int_0^1 g(u(1-u))du = \int_0^1 \frac{g(t/4)\,dt}{2(1-t)^{1/2}}. \tag{17}$$

If f is *symmetric*, meaning $f(u) = f(1-u)$, then $f(u)$ is a function of $u(1-u)$, and so is $u^{a-1}(1-u)^{a-1}f(u)$; the Euler integral (1) for $a = b$ can then be simplified by application of (17). As indicated in [13, Ex. II.9.2], for $a > 0$ the case of (17) with $g(x) = x^{a-1}$ yields

$$B(a,a) = 2^{1-2a}B(a, \tfrac{1}{2}) \tag{18}$$

which in view of Euler's formula $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ amounts to Legendre's duplication formula for the gamma function

$$\frac{\Gamma(2a)}{\Gamma(a)} = 2^{2a-1}\frac{\Gamma(a+\frac{1}{2})}{\Gamma(\frac{1}{2})}. \tag{19}$$

The table of Euler integrals in Exton [11] provides dozens of other examples of (17).

Proof of formula (9). Since the coefficient of y^k on each side of (9) is evidently a rational function of α and β, it suffices to establish the identity for $\alpha > 0$ and $\beta > 0$. Since

$$(1-yu)\left(1-\frac{y}{y-1}\,u\right) = 1-\frac{y^2}{y-1}\,u(1-u)$$

for $a = b = \alpha, p = q = \beta, z = y/(y-1)$, formula (7) reduces to

$$F_1\left(\alpha,\beta,\beta,2\alpha,y,\frac{y}{y-1}\right) = \frac{1}{B(\alpha,\alpha)}\int_0^1 \frac{[u(1-u)]^{\alpha-1}\,du}{[1-y^2\,u(1-u)/(y-1)]^{\beta}}$$

which equals the right side of (9) by application of (17), (18), and the integral representation (3) of Gauss's hypergeometric function. $\square$

In §6 we will further explore the symmetry or antisymmetry of functions around the middle of the domain of integration.

A duplication formula for a Lauricella function. The Lauricella function [18]

$$F_D^{(n)}(a,b_1,\ldots,b_n;c,x_1,\ldots,x_n) = \sum_{(m_1,\ldots,m_n)} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}x_1^{m_1}\cdots x_n^{m_n}}{(c)_{m_1+\cdots+m_n}\,m_1!\cdots m_n!}$$

where the sum is over all vectors of n non-negative integers $(m_1,\ldots,m_n)$, is known [10, (2.3.6)] to admit the integral representation

$$F_D^{(n)}(a,b_1,\ldots,b_n;c,x_1,\ldots,x_n) = \frac{1}{B(a,c-a)}\int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}du}{(1-ux_1)^{b_1}\cdots(1-ux_n)^{b_n}} \tag{20}$$

provided the real parts of a and $c-a$ are positive and $x_1,\ldots,x_n$ are in the open unit disc. Another application of (17) and (18) yields the following duplication formula, which reduces a Lauricella function of $2n$ variables, with $c = 2a$, with n equal pairs of parameters, and n corresponding pairs of variables x_i and $\hat{x}_i$ with $\hat{x}_i = x_i/(x_i-1)$ for $1 \le i \le n$, to a Lauricella function of n variables:

$$F_D^{(2n)}(a,b_1,b_1,\ldots,b_n,b_n;2a,x_1,\hat{x}_1,\ldots,x_n,\hat{x}_n) = F_D^{(n)}(a,b_1,\ldots,b_n;a+\tfrac{1}{2},y_1,\ldots,y_n) \tag{21}$$

where $y_i := x_i^2/(4x_i-4)$. See also Karlsson [17] for some reductions of generalized Kampé de Fériet functions obtained by a similar method.

3 Formula (10) and related topics

We first start with Gessel's proof of formula (11). Gessel argues that formula (11) can be obtained by taking the average of the following two formulas:

$$F_1(\alpha;\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2};2\alpha+1;y,z) = \left(\frac{2}{\sqrt{1-y}+\sqrt{1-z}}\right)^{2\alpha} \tag{22}$$

and

$$F_1(\alpha+1;\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2};2\alpha+1;y,z)=\frac{1}{\sqrt{1-y}\sqrt{1-z}}\left(\frac{2}{\sqrt{1-y}+\sqrt{1-z}}\right)^{2\alpha}. \tag{23}$$

To prove these formulas, apply the reduction formula [6, 9.5 (2)],

$$F_1(a;b,c;b+c;y,z)=(1-z)^{-a}\,F\left(\begin{matrix}a,b\\ b+c\end{matrix}\,\middle|\,\frac{y-z}{1-z}\right)$$

to get

$$F_1(\alpha;\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2};2\alpha+1;y,z)=(1-z)^{-\alpha}\,F\left(\begin{matrix}\alpha,\alpha+\frac{1}{2}\\ 2\alpha+1\end{matrix}\,\middle|\,\frac{y-z}{1-z}\right)$$

and

$$F_1(\alpha+1;\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2};2\alpha+1;y,z)=(1-z)^{-\alpha-1}\,F\left(\begin{matrix}\alpha+1,\alpha+\frac{1}{2}\\ 2\alpha+1\end{matrix}\,\middle|\,\frac{y-z}{1-z}\right)$$

Finally, apply the formulas [1, (15.1.13) and (15.1.14), p. 556]

$$F\left(\begin{matrix}\alpha,\alpha+\frac{1}{2}\\ 2\alpha+1\end{matrix}\,\middle|\,u\right)=\left(\frac{2}{1+\sqrt{1-u}}\right)^{2\alpha} \tag{24}$$

and

$$F\left(\begin{matrix}\alpha+1,\alpha+\frac{1}{2}\\ 2\alpha+1\end{matrix}\,\middle|\,u\right)=\frac{1}{\sqrt{1-u}}\left(\frac{2}{1+\sqrt{1-u}}\right)^{2\alpha} \tag{25}$$

and simplify.

To discover what is behind formula (11) we appeal to (2), p. 239 in [8] and get, for $j=0,1,2,\ldots$, the following result

$$\begin{aligned}&(1-y)^{\alpha}F_1(\alpha,\gamma+j-\beta',\beta',\gamma;y,z)\\ &\quad=F_1\left(\alpha,-j,\beta',\gamma;\frac{y}{y-1},\frac{z-y}{1-y}\right)\\ &\quad=\sum_{m=0}^{j}\frac{(-j)_m}{m!}\left(\frac{y}{y-1}\right)^m\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta')_n}{(\gamma)_{m+n}\,n!}\left(\frac{z-y}{1-y}\right)^n.\end{aligned}$$

Therefore

$$\begin{aligned}F_1(\alpha,\gamma+j-\beta',\beta',\gamma;y,z)\;&=\;(1-y)^{-\alpha}\sum_{m=0}^{j}\frac{(-j)_m(\alpha)_m}{m!(\gamma)_m}\left(\frac{y}{y-1}\right)^m\\ &\qquad\times F\left(\begin{matrix}\alpha+m,\beta'\\ \gamma+m\end{matrix}\,\middle|\,\frac{z-y}{1-y}\right).\end{aligned} \tag{26}$$

At this stage one needs to make judicious choices of the parameters of the hypergeometric function on the right-hand side of formula (26) to reduce it to an algebraic function. One choice is

$$\gamma=\alpha+\beta'-1/2. \tag{27}$$

When $j=1$ the right-hand side of (26) is the sum of the F in (25) and a similar F with a different value of α. This explains Gessel's identity.

For general j we chose $\beta'=\alpha+1/2$ then apply the quadratic transformation [8, (2.11.13)]. The result is that for all $j=0,1,2,\ldots$ and $\beta'=\alpha+1/2$ the F in (26) is

$$\left(\frac{1-z}{1-y}\right)^{1/2} F\left(\left.\begin{matrix} 2\alpha+2m-1, 2\alpha \\ 2\alpha+m \end{matrix}\right| \frac{1}{2}-\frac{1}{2}\left(\frac{1-z}{1-y}\right)^{1/2}\right).$$

The hypergeometric function in the above expression is algebraic as can be seen from the Pfaff-Kummer transformation (6). The final result is

$$\begin{aligned} &F_1(\alpha,\alpha+j-\beta',\beta',\gamma;y,z) \\ &= \left[\sqrt{1-y}+\sqrt{1-z}\right]^{-2\alpha} 2^{2\alpha}\sqrt{\frac{1-z}{1-y}}\sum_{m=0}^{j}\sum_{k=0}^{m+1}\frac{(-j)_m(\alpha)_m(-m-1)_k}{m!k!(2\alpha+k)_m}\left(\frac{y}{y-1}\right)^m \\ &\quad\times(y-z)^k\left[\sqrt{1-y}+\sqrt{1-z}\right]^{-2k}. \end{aligned} \tag{28}$$

Another case is

$$\gamma=\alpha-\beta'+1. \tag{29}$$

We need to apply

$$F\left(\left.\begin{matrix} a, \quad b \\ a-b+1 \end{matrix}\right| z\right) = (1+z)^{-a} F\left(\left.\begin{matrix} a/2,(a+1)/2 \\ a-b+1 \end{matrix}\right| \frac{4z}{(1+z)^2}\right), \tag{30}$$

[8, (2.11.34)]. The quadratic transformation (29) had a misprint in the original reference where $a-b+1$ on the right-hand side was printed as $a-b-1$. In this case, that is when (29) holds, (29) becomes

$$\begin{aligned} &F\left(\left.\begin{matrix} \alpha+m, \quad \beta' \\ \alpha-\beta'+1+m \end{matrix}\right| \frac{z-y}{1-y}\right) \\ &= \left(\frac{1-y}{1-2y+z}\right)^{\alpha+m} F\left(\left.\begin{matrix} (\alpha+m)/2,(\alpha+m+1)/2 \\ \alpha-\beta'+1+m \end{matrix}\right| \frac{4(z-y)(1-y)}{(1-2y+z)^2}\right). \end{aligned}$$

Now (26) becomes

$$\begin{aligned} &F_1(\alpha,\alpha+1+j-\beta',\alpha-\beta'+1,y,z) \\ &= \sum_{m=0}^{j}\frac{(-j)_m(\alpha)_m}{m!(\alpha-\beta'+1)_m}\frac{(-y)^m}{(1-2y+z)^{\alpha+m}} \\ &\quad\times F\left(\left.\begin{matrix} (\alpha+m)/2,(\alpha+m+1)/2 \\ \alpha-\beta'+1+m \end{matrix}\right| \frac{4(z-y)(1-y)}{(1-2y+z)^2}\right). \end{aligned} \tag{31}$$

It is clear from this formula that the choice $\beta'=k+1+\alpha/2$, so that $\gamma=-k+\alpha/2$, $k=1,2,\ldots$, $k\geq(j+1)/2$, and the Pfaff-Kummer transformation (6) reduce the F_1 in (30) to an algebraic function.

We take this opportunity to add that a general useful identity which generalizes (6) is the Fields and Wimp formula [15]

$$
\begin{aligned}
&{}_{p+r}F_{q+s}\left(\begin{matrix} a_1, \ldots, a_p, & c_1, \ldots, c_r \\ b_1, \ldots, b_q, & d_1, \ldots, d_s \end{matrix} \middle| \, zw \right) \\
&\quad = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{(-z)^n}{n!} \; {}_pF_q\left(\begin{matrix} n+a_1, \ldots, n+a_p \\ n+b_1, \ldots, n+b_q \end{matrix} \middle| \, z \right) \\
&\qquad \times \; {}_{r+1}F_s\left(\begin{matrix} -n, c_1, \ldots, c_r \\ d_1, \ldots, d_s \end{matrix} \middle| \, w \right).
\end{aligned} \tag{32}
$$

For a treatment of such formulas, see [14]. One can apply (31) istead of (6) to functions of several variables defined by beta type integrals like (7) and establish similar identities. We shall not discuss this any further in this work.

4 Brownian variations.

Let $(b(u), 0 \le u \le 1)$ be a *standard Brownian bridge*, that is the centered Gaussian process with continuous sample paths and covariance function

$$E(b(u)b(v)) = u(1-v) \quad (0 \le u \le v \le 1), \tag{33}$$

obtained by conditioning a standard one-dimensional Brownian motion started at 0 to return to 0 at time 1. See [27, 26] for background. Let V_n denote the variation of the path of b over the random partition of $[0,1]$ defined by cutting the interval at each of n points picked uniformly at random from $[0,1]$, independently of each other and of b. That is

$$V_n := \sum_{i=1}^{n+1} |\widetilde{\Delta}_{n,i}| \qquad \text{where } \widetilde{\Delta}_{n,i} := b(U_{n,i}) - b(U_{n,i-1}) \tag{34}$$

for $U_{n,1} < U_{n,2} < \cdots < U_{n,n}$ the *uniform order statistics* obtained by putting the n uniform random variables in increasing order, and $U_{n,0} = 0, U_{n,n+1} = 1$. In [24] the distribution of V_n is characterized as the unique distribution on $(0,\infty)$ whose pth moment is given for each $p > 0$ by the formula

$$EV_n^p = 2^{p/2} \frac{\Gamma(n+p)}{\Gamma(n)} \frac{\Gamma(2n)}{\Gamma(2n+p)} \frac{\Gamma(n+p/2)}{\Gamma(n)} \tag{35}$$

and the corresponding density is expressed in terms of the Hermite function defined in [19]. In particular, (35) gives

$$EV_n = \frac{1}{\sqrt{2}} \frac{\Gamma(n+1/2)}{\Gamma(n)}; \quad EV_n^2 = \frac{n(n+1)}{2n+1}. \tag{36}$$

The formula for EV_n is easily checked as follows, by conditioning on the U_{ni}. It is well known that the Brownian bridge b has exchangeable increments, and that the spacings $\Delta_{n,i} :=$

$U_{n,i} - U_{n,i-1}$ for $1 \le i \le n+1$ are exchangeable [2]. It follows that in the sum (34) defining V_n the $n+1$ terms $|\widetilde{\Delta}_{n,i}|$ are exchangeable. Combined with the consequence of (33) that

$$\widetilde{\Delta}_{n,1} \stackrel{d}{=} \sqrt{U_{n,1}\bar{U}_{n,1}}\, Z \tag{37}$$

where $\bar{U}_{n,1} := 1 - U_{n,1}$, and Z is a standard Gaussian variable independent of $U_{n,1}$, so

$$E|Z| = \sqrt{\frac{2}{\pi}}, \qquad EZ^2 = 1, \tag{38}$$

the exchangeability of the $|\widetilde{\Delta}_{n,i}|$ allows the following evaluation:

$$\begin{aligned} EV_n &= (n+1)E|\widetilde{\Delta}_{n,1}| = (n+1)\left(E\sqrt{U_{n,1}\bar{U}_{n,1}}\right)E|Z| \\ &= (n+1)n\frac{\Gamma(\frac{3}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+2)}\sqrt{\frac{2}{\pi}} \end{aligned}$$

which reduces to the expression for EV_n in (36).

It is not so easy to check the evaluation of EV_n^2 in (36) by the same method. Rather, this method yields an expression for EV_n^2 which when compared with that in (36) leads by a remarkable sequence of integral identities to the evaluation of Euler integral presented in the introduction as (1), an example is (57). It might also be interesting to explore the integral identities implied similarly by (35) for $n = 3, 4, \ldots$, but these appear to be much more complicated.

By the same considerations of exchangeability

$$EV_n^2 = E\left(\sum_{i=1}^{n+1}|\widetilde{\Delta}_{n,i}|\right)^2 = (n+1)E\widetilde{\Delta}_{n,1}^2 + (n+1)nE|\widetilde{\Delta}_{n,1}\widetilde{\Delta}_{n,2}|. \tag{39}$$

Now by (37)

$$(n+1)E\widetilde{\Delta}_{n,1}^2 = (n+1)\left(E(U_{n,1}\bar{U}_{n,1})\right)EZ^2 = (n+1)\frac{n}{(n+1)(n+2)} = \frac{n}{n+2} \tag{40}$$

and by a straightforward extension of (37)

$$E|\widetilde{\Delta}_{n,1}\widetilde{\Delta}_{n,2}| = E\left(|X_n|\sqrt{\Delta_{n,1}\bar{\Delta}_{n,1}}\,|Y_n|\sqrt{\Delta_{n,2}\bar{\Delta}_{n,2}}\right) \tag{41}$$

where $\bar{x} := 1 - x$ and (X_n, Y_n) is a pair of random variables which given $\Delta_{n,1}$ and $\Delta_{n,2}$ has the standard bivariate normal distribution with correlation

$$E(X_nY_n \mid \Delta_{n,1}, \Delta_{n,2}) := -\sqrt{\frac{\Delta_{n,1}\Delta_{n,2}}{\bar{\Delta}_{n,1}\bar{\Delta}_{n,2}}}. \tag{42}$$

Here $\Delta_{n,1}$ and $\Delta_{n,2}$ have the same joint distribution as $\min_{1\le i\le n} U_i$ and $1 - \max_{1\le i\le n} U_i$ for independent uniform $(0,1)$ variables U_i. For $n \ge 2$ this means

$$P(\Delta_{n,1} \in dx, \Delta_{n,2} \in dy) = n(n-1)(1-x-y)^{n-2}dx\,dy \qquad (x, y \ge 0, x+y \le 1). \tag{43}$$

The expectation in (41) can be evaluated with the help of the following lemma:

Lemma 1 *Let (X,Y) have the standard bivariate normal distribution with*

$$E(X) = E(Y) = 0; \quad E(X^2) = E(Y^2) = 1; \quad E(XY) = r \in [-1,1],$$

meaning that

$$Y = rX + \sqrt{1-r^2}Z \tag{44}$$

for independent standard normal X and Z. Then

$$E|XY| = \frac{2}{\pi}\left(\sqrt{1-r^2} + r \arcsin r\right). \tag{45}$$

Proof. By conditioning on X and applying a standard integral representation of the McDonald function K_0, as indicated in [29], there is the following correction of formula (2) of [29]: for all real z

$$P(XY \in dz) = \frac{1}{\pi\sqrt{1-r^2}} \exp\left(\frac{rz}{1-r^2}\right) K_0\left(\frac{|z|}{1-r^2}\right) dz. \tag{46}$$

The classical identity

$$\int_0^\infty e^{az} K_0(bz) = \frac{1}{\sqrt{b^2-a^2}}\left(\frac{\pi}{2} - \arcsin\frac{a}{b}\right) \quad (0 \le |a| < b) \tag{47}$$

now yields the moment generating function of $|XY|$: for $|\theta| < (1+r)^{-1}$

$$Ee^{\theta|XY|} = \frac{\pi + 2\arcsin(r + \theta(1-r^2))}{2\pi\sqrt{1 - r\theta - (1-r^2)\theta^2}} + \frac{\pi - 2\arcsin(r - \theta(1-r^2))}{2\pi\sqrt{1 + r\theta - (1-r^2)\theta^2}} \tag{48}$$

and (45) is read from the coefficient of θ in the expansion of (48) in powers of θ. $\square$

As a check on (47), this formula combined with (46) yields the following companion of (45), which is a well known consequence of (44) and the symmetry of the joint distribution of (X,Z) under rotations:

$$P(XY > 0) = \frac{1}{2} + \frac{1}{\pi}\arcsin r$$

As checks on (48), the coefficient of θ^2 agrees with the formula $E(XY)^2 = 1 + 2r^2$ which is obvious from (44), and the coefficients of θ^n for small even n are found to agree with those of the well known m.g.f. of XY:

$$Ee^{\theta XY} = 1/\sqrt{1 - 2r\theta - (1-r^2)\theta^2}. \tag{49}$$

The computation in (41) can now be continued by conditioning on $(\Delta_{n,1}, \Delta_{n,2})$ and applying (42) and (45) to evaluate

$$E|\widetilde{\Delta}_{n,1}\widetilde{\Delta}_{n,2}| = \frac{2}{\pi}E\sqrt{\Delta_{n,1}\Delta_{n,2}(\overline{\Delta_{n,1} + \Delta_{n,2}})} + \frac{2}{\pi}E\left(\Delta_{n,1}\Delta_{n,2}\arcsin\sqrt{\frac{\Delta_{n,1}\Delta_{n,2}}{\bar{\Delta}_{n,1}\bar{\Delta}_{n,2}}}\right) \tag{50}$$

where $\bar{x} := 1 - x$. By (43), the first term is a Dirichlet integral, which is easily evaluated as $\frac{1}{2}(n-1)_2/(n-\frac{1}{2})_3$, where $(\theta)_m$ is the rising factorial with m factors. Substitute this in (50), then (50) and (40) in (39), and compare with (36) to deduce that for each $n = 1, 2, \ldots$

$$\frac{2}{\pi} E\left(\Delta_{n,1}\Delta_{n,2} \arcsin \sqrt{\frac{\Delta_{n,1}\Delta_{n,2}}{\bar{\Delta}_{n,1}\bar{\Delta}_{n,2}}}\right) = \frac{3(3n^2+3n-1)}{8(n+1)_2(n-\frac{1}{2})_3}. \tag{51}$$

If $n = 1$ then $\Delta_{1,1} = \bar{\Delta}_{1,2} = U$ say has uniform distribution on $(0,1)$, and (51) reduces to the elementary evaluation $E(U\bar{U}) = 1/6$. For $n \geq 2$ set $p = n - 2$. In view of (43), the identity (51) with both sides divided by $n(n-1) = (p+1)_2$ reads

$$\frac{2}{\pi} \int\!\!\int_{x,y\geq 0, x+y\leq 1} xy(\overline{x+y})^p \arcsin \sqrt{\frac{xy}{\bar{x}\bar{y}}}\, dx\, dy = \frac{3(3p^2+15p+17)}{8(p+1)_4(p+\frac{3}{2})_3}. \tag{52}$$

The following argument shows that this identity holds in fact for all complex p with positive real part. The left side of (52) is evidently the pth moment $\int_0^1 z^p f(z)dz$ of a positive density f on $(0,1)$, which is found by partial fraction expansion of the right side of (52) to be

$$f(z) = \frac{1}{12}(1-\sqrt{z})^4(2+\sqrt{z})(1+2\sqrt{z}). \tag{53}$$

In view of (43) for $n = 2$ and its consequence that $\Delta_{2,3} = \overline{\Delta_{2,1}+\Delta_{2,2}}$ has probability density $2(1-z)$ at $z \in (0,1)$, it follows from the above discussion that the function $f(z)/(1-z)$ can be interpreted as follows as a conditional expectation: for $0 < z < 1$

$$\frac{2}{\pi} E\left(\Delta_{2,1}\Delta_{2,2} \arcsin \sqrt{\frac{\Delta_{2,1}\Delta_{2,2}}{\bar{\Delta}_{2,1}\bar{\Delta}_{2,2}}}\,\middle|\, \Delta_{2,3} = z\right) = \frac{f(z)}{(1-z)}. \tag{54}$$

Since $\Delta_{2,1}$ given $\Delta_{2,3} = z$ has uniform distribution on $1 - z$, which is the distribution of $U(1-z)$ for U with uniform distribution on $(0,1)$, it follows after setting $w = (1-z)$ and $\bar{U} = (1-U)$ that for $0 < w \leq 1$

$$\frac{2}{\pi} E\left(U\bar{U} \arcsin \sqrt{\frac{w^2 U\bar{U}}{(1-wU)(1-w\bar{U})}}\right) = \frac{f(1-w)}{w^3}. \tag{55}$$

5 Some symmetric Euler integrals.

Substitute $x = 1/w$ in (55), use the formula (53) for f, and simplify, to see that (55) amounts to the following identity: for all $x \geq 1$

$$\int_0^1 u\bar{u} \arcsin \sqrt{\frac{u\bar{u}}{(x-u)(x-\bar{u})}}\, du = \frac{\pi}{24}\left(-8x^3+12x^2-2+\sqrt{x(x-1)}(8x^2-8x-3)\right) \tag{56}$$

The formula

$$\arcsin \sqrt{1/(1+z)} = \pi/2 - \arctan \sqrt{z}$$

reduces (56) to

$$\int_0^1 u(1-u)\arctan\sqrt{\frac{x(x-1)}{u(1-u)}}du = \frac{\pi}{24}\left(8x^3-12x^2+4-\sqrt{x(x-1)}(8x^2-8x-3)\right). \quad (57)$$

Equivalently, from (57) via (17),

$$\int_0^1 \frac{t}{\sqrt{t}}\arctan\sqrt{\frac{4x(x-1)}{t}}dt = \frac{\pi}{3}\left(8x^3-12x^2+4-\sqrt{x(x-1)}(8x^2-8x-3)\right) \quad (58)$$

which can be verified using *Mathematica*. To relate (57) to the identities discussed in the introduction, consider for $x \geq 1, \alpha > 0$ and arbitrary real β the integral

$$I(\alpha,\beta;x) := \int_0^1 \frac{(u(1-u))^{\alpha-1}}{((x-u)(x-1+u))^{\beta}}du \quad (59)$$

where the denominator of the integrand can be expressed differently using

$$(x-u)(x-1+u) = x(x-1)+u(1-u) = x(x-1)\left(1-\frac{u}{x}\right)\left(1-\frac{u}{1-x}\right). \quad (60)$$

By differentiating (57) with respect to x, and dividing both sides by $2x-1$, formula (57) is seen to be equivalent to

$$I(\tfrac{5}{2},1;x) = \frac{2\pi x^{3/2}(x-1)^{3/2}}{2x-1} - \frac{\pi}{8}(8x^2-8x-1). \quad (61)$$

It is easily seen that

$$I(\alpha,\beta+1;x) = \frac{-1}{\beta(2x-1)}\frac{d}{dx}I(\alpha,\beta;x) \quad (62)$$

and provided $\beta > 0$ this operation is inverted by

$$I(\alpha,\beta;x) = \beta\int_x^\infty (2y-1)I(\alpha,\beta+1,;y)dy. \quad (63)$$

Thus by repeatedly differentiating and dividing by $2x-1$, we find that (61) is equivalent in turn to each of an infinite chain of explicit evaluations of $I(\frac{5}{2},\beta;x)$ for $\beta = 1,2,\ldots$, the next few of which are

$$I(\tfrac{5}{2},2;x) = \pi\left(1-\frac{\sqrt{x(x-1)}(8x^2-8x+3)}{(2x-1)^3}\right) \quad (64)$$

$$I(\tfrac{5}{2},3;x) = \frac{3\pi}{4\sqrt{x(x-1)}(2x-1)^5} \quad (65)$$

$$I(\tfrac{5}{2},4;x) = \frac{\pi(24x^2-24x+1)}{8x^{3/2}(x-1)^{3/2}(2x-1)^7} \quad (66)$$

and so on. The simplest of this sequence of identities is (65). This is the special case $\alpha = \frac{5}{2}$ of the following identity: for all real $\alpha > 0$ and $x \geq 1$

$$I(\alpha, \alpha + \tfrac{1}{2}; x) = \frac{B(\alpha,\alpha)2^{2\alpha}}{\sqrt{x(x-1)}(2x-1)^{2\alpha}}. \tag{67}$$

By using the last expression in (60) and (7) the integral $I(\alpha, \beta; x)$ can be presented as

$$I(\alpha,\beta;x) = \frac{B(\alpha,\alpha)}{x^\beta (x-1)^\beta} F_1\left(\alpha,\beta,\beta,2\alpha; \frac{1}{x}, \frac{1}{1-x}\right) \tag{68}$$

where F_1 is Appell's hypergeometric function (8). Thus (67) is just a restatement of (10), and (9) amounts to the following more general evaluation: for $x \geq 1, \alpha > 0$ and real β,

$$I(\alpha,\beta;x) = \frac{B(\alpha,\alpha)}{x^\beta (x-1)^\beta} F\left(\begin{array}{c} \alpha, \beta \\ \alpha + \frac{1}{2} \end{array} \middle| \frac{-1}{4x(x-1)}\right). \tag{69}$$

The operation (62) shows that for each α formula (67) generates an infinite sequence of explicit evaluations of $I(\alpha, \alpha + d + \frac{1}{2}; x)$ in terms of algebraic functions for $d = 0, 1, 2, \ldots$. For some α, as in the case for $\alpha = \frac{5}{2}$ as illustrated above, it is also possible to work backwards using (63) to get algebraic expressions for $I(\alpha, \alpha + d + \frac{1}{2}; x)$ for negative d. Now if d is a positive integer the Gauss function on the right side of (5) is a polynomial in z. Substituted in (69) with $a = \alpha$ and $c = \alpha + \frac{1}{2}$, this gives an explicit formula for $I(\alpha, \alpha + d + \frac{1}{2}; x)$ instead of a recursive evaluation. The question of whether or not there is an explicit representation for $I(\alpha, \beta; x)$ in terms of elementary functions of x for a particular choice of (α, β) reduces to whether or not the Gauss function appearing in (69) can be so represented. See for [8, 1] for tabulations of such elementary evaluations of the Gauss function.

6 Generalized beta densities

For arbitrary positive a, b, and real c_j and y_j with $|y_j| < 1$, $1 \leq j \leq n$ the formula

$$u^{a-1}(1-u)^{b-1} \prod_{j=1}^{n} (1 - y_j u)^{-c_j} \tag{70}$$

defines a positive function of $u \in (0,1)$ which can be normalized to define a probability density on $(0,1)$, which we shall call a *generalized beta density*. As noted by Exton [10, §7.1.1], the integral representation (20) of the Lauricella function $F_D^{(n)}$ implies that this function appears in the normalization constant and in formulae for the moments of this family of distributions on $(0,1)$. The functions in (70) are weight functions for orthogonal polynomials which generalize Jacobi polynomials. These weight functions are referred to as *generalized Jacobi weights* [22].

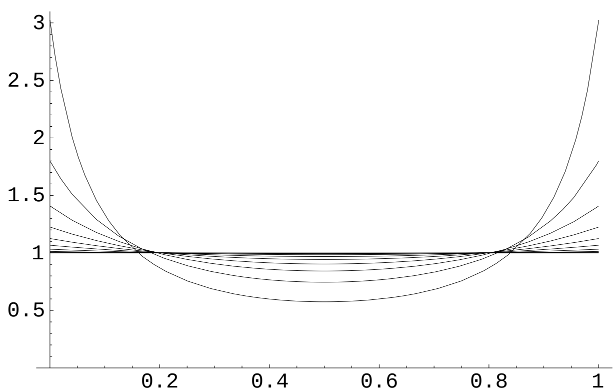
The following discussion concerns a particular one-parameter sub-family of this multi-parameter family of distributions on $(0,1)$, which is related to the duplication formula (9) for the Appell function F_1. For each $y \in [0,1)$ define non-negative functions Ψ_y and f_y with domain $[0,1]$ by the formulae

$$\Psi_y(u) := \frac{u(1-u)(1-\frac{1}{2}y)^2}{(1-yu)(1-y(1-u))} \text{ and } f_y(u) := \frac{(1-y)^{\frac{1}{2}}(1-\frac{1}{2}y)^2}{((1-yu)(1-y(1-u))^{\frac{3}{2}}}. \tag{71}$$

Observe that in view of (7), formula (10) can be rewritten as follows: for each $y \in [0,1)$ and $\alpha > 0$

$$\int_0^1 [\Psi_y(u)]^{\alpha-1} f_y(u)\, du = B(\alpha,\alpha) = \int_0^1 [u(1-u)]^{\alpha-1} du. \tag{72}$$

For $\alpha = 1$ this shows f_y is a probability density on $(0,1)$ for each $y \in [0,1)$. This family is evidently the subfamily of the family of densities obtained by normalization of the functions (70), for $a = b = 1$, $n = 2$, $y_1 = y$, $y_2 = y/(y-1)$ and $c_1 = c_2 = \frac{3}{2}$. The following graphs show the densities of f_y for $y = i/10, 0 \le i \le 9$.



The graphs illustrate the following facts which are easily verified by calculus. For each $0 < y < 1$ the density f_y is convex and symmetric about $1/2$, with maximum value $(1-\frac{1}{2}y)^2/(1-y)$ attained at 0 and 1, and as $y \uparrow 1$ the probability distribution with density f_y converges weakly to the Bernoulli($\frac{1}{2}$) distribution on $\{0,1\}$.

According to formula (72), if U_y denotes a random variable with density f_y, and $U = U_0$ is a random variable with uniform distribution on $(0,1)$, then

$$E[\Psi_y(U_y)]^{\alpha-1} = B(\alpha,\alpha) = E[U(1-U)]^{\alpha-1} \qquad (\alpha > 0) \tag{73}$$

Since a distribution on $(0,1)$ is uniquely determined by its moments, this implies the identity in distribution

$$\Psi_y(U_y) \stackrel{d}{=} U(1-U). \tag{74}$$

Equivalently, for each non-negative measurable function g with domain $[0,1/4]$

$$\int_0^1 g(\Psi_y(u))\, f_y(u)\, du = \int_0^1 g(u(1-u))du = \int_0^1 \frac{g(x/4)dx}{2(1-x)^{1/2}}. \tag{75}$$

where the second equality is just (17) again. This second equality shows that $4U(1-U)$ has the beta $(1,\frac{1}{2})$ distribution with density $\frac{1}{2}(1-u)^{-1/2}$ at $u \in (0,1)$, hence that $U(1-$

$U) \stackrel{d}{=} \frac{1}{4}(1-U^2)$. It then follows by inversion of the transformation Ψ_y that a random variable U_y with density f_y can be constructed as

$$U_y := \tfrac{1}{2} \pm \tfrac{1}{2} U \left(1 - \frac{y^2}{(2-y)^2}(1-U^2) \right)^{-1/2} \tag{76}$$

where $\pm$ is a random sign, equally likely to be $+1$ or -1, independent of U. By symmetry, U_y has mean $1/2$ for all y. The variance of U_y is found by integration using (76) to be

$$E(U_y - \tfrac{1}{2})^2 = \frac{(2-y)^2}{4y^2} \left(1 - 2\frac{\sqrt{1-y}}{y} \arctan \frac{y}{2\sqrt{1-y}} \right). \tag{77}$$

As the first equality in (75) does not seem obvious, we provide the following check:
Direct proof of the first equality in (75). Since $\Psi_y(u) = \Psi_y(1-u)$ and $f_y(u) = f_y(1-u)$, the left side of (71) equals

$$2\int_0^{\frac{1}{2}} g(\Psi_y(\tfrac{1}{2}+v))\, f_y(\tfrac{1}{2}+v)\, dv = 2\int_0^{\frac{1}{4}} g(w)\, f_y(\tfrac{1}{2}+v_y(w)) \left| \frac{dv_y(w)}{dw} \right| dw \tag{78}$$

by the change of variable $w = \Psi_y(\frac{1}{2}+v)$ which makes w decrease from $1/4$ to 0 as v increases from 0 to $\frac{1}{2}$, with

$$v = v_y(w) := \frac{1}{2}\sqrt{\frac{1-4w}{1-c_y w}}, \qquad \frac{dv_y(w)}{dw} = \frac{c_y - 4}{4(1-4w)^{\frac{1}{2}}(1-c_y w)^{\frac{3}{2}}}$$

where

$$c_y := \frac{y^2}{(1-\frac{1}{2}y)^2}.$$

The right side of (78) can be simplified, by use of the following identities, which follow easily from the above definitions:

$$f_y(\tfrac{1}{2}+v) = \frac{2(2-y)^2(1-y)^{\frac{1}{2}}}{((2-y)^2 + 4v^2y^2)^{\frac{3}{2}}}$$

where for $v = v_y(w)$

$$(2-y)^2 + 4v^2y^2 = (2-y)^2 + \frac{(1-4w)y^2}{1-c_y w} = \frac{4(1-y)}{1-c_y w}$$

so

$$f_y(\tfrac{1}{2}+v_y(w)) = \frac{2(2-y)^2(1-y)^{\frac{1}{2}}(1-c_y w)^{\frac{3}{2}}}{4^{\frac{3}{2}}(1-y)^{\frac{3}{2}}} = \frac{(2-y)^2(1-c_y w)^{\frac{3}{2}}}{4(1-y)} \tag{79}$$

and $4 - c_y = 16(1-y)/(2-y)^2$ so that

$$\left| \frac{dv_y(w)}{dw} \right| = \frac{16(1-y)}{4(2-y)^2(1-4w)^{\frac{1}{2}}(1-c_y w)^{\frac{3}{2}}} \tag{80}$$

and the equality of first and last expressions in (75) follows by cancellation, and a final change of variable $x = 4w$. $\square$

7 Companion identities

For arbitrary non-negative measurable functions h and g there is the following extension of the change of variable formula (17) to deal with asymmetric integrands, obtained by setting $t = 4u(1-u)$ so $u = \frac{1}{2}(1 \pm \sqrt{1-t}), |du/dt| = 1/(4\sqrt{1-t})$:

$$\int_0^1 h(u(1-u))g(u)du = \sum_{\sigma \in \{\pm 1\}} \int_0^1 \frac{h(t/4)g(\frac{1}{2}(1+\sigma\sqrt{1-t}))}{4\sqrt{1-t}} dt. \tag{81}$$

We now show how (11) and other related formulas follow from (81). Here again we prove a more general result which may be of independent interest. The basic ingredients are the following relationships involving the ultraspherical polynomials $\{C_n^\nu(x)\}$ and Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$,

$$C_{2m}^\nu(x) = \frac{(\nu)_m}{(1/2)_m} P_m^{(\nu-1/2,-1/2)}(2x^2-1), \tag{82}$$

$$C_{2m+1}^\nu(x) = \frac{(\nu)_{m+1}}{(1/2)_{m+1}} x P_m^{(\nu-1/2,1/2)}(2x^2-1), \tag{83}$$

$$\sum_{n=0}^\infty P_n^{(\alpha,\beta)}(\zeta)w^n = 2^{\alpha+\beta}R^{-1}(1-w+R)^{-\alpha}(1+w+R)^{-\beta}, \tag{84}$$

where

$$R = (1-2\zeta w + w^2)^{1/2}. \tag{85}$$

See for example (10.9.21), (10.9.22), (10.9.29), and (10.9.30), respectively in [9]. We shall also need the well-known generating function

$$\sum_{n=0}^\infty C_n^\nu(y)w^n = (1-2yw+w^2)^{-\nu}. \tag{86}$$

Theorem 2 *We have*

$$F_1(\alpha,\beta,\beta,2\alpha;y,z) = [(1-y/2)(1-z/2)]^{-\beta} \times \sum_{n=0}^\infty \frac{(yz)^n}{[(2-y)(2-z)]^n} \frac{(\beta)_n}{(\alpha+1/2)_n} P_n^{(\beta-1/2,-1/2)}(Z), \tag{87}$$

with Z defined as

$$Z = \frac{2(y+z-yz)^2}{yz(2-y)(2-z)} - 1, \tag{88}$$

and

$$F_1(\alpha+1,\beta,\beta,2\alpha+1;y,z) - F_1(\alpha,\beta,\beta,2\alpha;y,z) = \frac{4^\beta(y+z-yz)}{[(2-y(2-z)]^{\beta+1}} \sum_{m=0}^\infty \frac{(yz)^m(\beta)_{m+1}}{[(2-y)(2-z)]^m(\alpha+1/2)_{m+1}} P_m^{(\beta-1/2,1/2)}(Z) \tag{89}$$

Proof. Substitution of the right-hand side of (86) in the integral representation (7) ([9, (5.8.5)]) shows that the left-hand side of (87) is

$$
\begin{aligned}
&\frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \int_0^1 [u(1-u)]^{\alpha-1} \left[1 - \left(\frac{y}{2-y} + \frac{z}{2-z}\right)(2u-1) + \frac{yz(2u-1)^2}{(2-y)(2-z)}\right]^{-\beta} du \\
&= \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \int_0^1 \left(\frac{t}{4}\right)^{\alpha-1} \frac{1}{4\sqrt{1-t}} \\
&\quad \times 2 \sum_{n\ge 0, n \text{ even}} \frac{(yz)^{n/2}}{[(2-y)(2-z)]^{n/2}} C_n^\beta\left(\frac{y+z-yz}{\sqrt{yz(2-y)(2-z)}}\right)(1-t)^{n/2}\, dt.
\end{aligned}
$$

The above expression then simplifies to the right-hand side of (87) through the use of (83). Using the integral representiation (7) and the duplication formula for the gamma function, the difference in (89) is seen to equal

$$
\frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \int_0^1 [u(1-u)]^{\alpha-1}(2u-1)[(1-uy)(1-uz)]^{-\beta}\, du
$$

$$
\begin{aligned}
&= \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} \int_0^1 \left(\frac{t}{4}\right)^{\alpha-1} \frac{1}{4\sqrt{1-t}} \times 2\sqrt{1-t} \\
&\quad \times \sum_{n\ge 0, n \text{ odd}} \frac{(yz)^{n/2}}{[(2-y)(2-z)]^{n/2}} C_n^\beta\left(\frac{y+z-yz}{\sqrt{yz(2-y)(2-z)}}\right)(1-t)^{n/2}\, dt. \qquad (90)
\end{aligned}
$$

by expansion of the integrand as in the previous case. This simplifies via (83) to the expression in (89). □

When $\beta = \alpha + 1/2$ in Theorem 2, the generating function (84)–(85) implies (11). Another special case is to choose $z = y/(y-1)$ but keep β general subject to restrictions that make the integrals and sums involved converge. This choice makes $y + z - yz = 0$ and hence $Z = -1$. Now [9, (10.8.3), (10.8.13)] imply

$$
P_n^{(\alpha,\beta)}(-1) = \frac{(\beta+1)_n}{n!}(-1)^n.
$$

Thus we get

$$
F_1\left(\alpha, \beta, \beta, 2\alpha; y, \frac{y}{y-1}\right) = \frac{(1-y)^\beta}{(1-y/2)^{2\beta}}\, F\left(\left.\begin{matrix}\beta, 1/2\\ \alpha+\frac{1}{2}\end{matrix}\right| \frac{y^2}{(2-y)^2}\right) \qquad (91)
$$

which is equivalent to (9) through the Pfaff-Kummer transformation (6).

The same cases $\beta = \alpha + 1/2$ or $z = y/(y-1)$ are of interest in the second formula of Theorem 2. Thus

$$
F_1(\alpha+1, \alpha+\tfrac{1}{2}, \alpha+\tfrac{1}{2}, 2\alpha+1; y, z)
$$

$$= \frac{1}{2}\left(1+\frac{1}{\sqrt{1-y}\sqrt{1-z}}\right)\left(\frac{2}{\sqrt{1-y}+\sqrt{1-z}}\right)^{2\alpha}$$
$$+\frac{2^{2\alpha+1}(y+z-yz)}{[(2-y(2-z)]^{\alpha+3/2}}\sum_{m=0}^{\infty}\frac{(yz)^m(\beta)_{m+1}}{[(2-y)(2-z)]^m(\alpha+1/2)_{m+1}}P_m^{(\alpha,1/2)}(Z). \tag{92}$$

In the notation of (88),

$$\zeta = Z, \qquad R=\frac{4\sqrt{(1-y)(1-z)}}{(2-y)(2-z)}, \qquad w=\frac{yz}{(2-y)(2-z)}$$
$$1-w+R = 2\frac{\left(\sqrt{1-y}+\sqrt{1-z}\right)^2}{(2-y)(2-z)}, \quad 1+w+R=2\frac{\left[1+\sqrt{(1-y)(1-z)}\right]^2}{(2-y)(2-z)}.$$

Now (92) implies formula (23).

Another way to evaluate the integral on the left-hand side of (89) as

$$\tfrac{1}{2}[F_1(\alpha+1,\beta,\beta,2\alpha+1;y,z)-F_1(\alpha,\beta,\beta,2\alpha+1;y,z)].$$

This establishes formula (22). Finally the case $z=y/(y-1)$ in (89) gives the identity

$$F_1(\alpha+1,\beta,\beta,2\alpha+1;y,y/(y-1)) = F_1(\alpha,\beta,\beta,2\alpha;y,y/(y-1)). \tag{93}$$

Thus (9) can be recast as

$$F_1(\alpha+1,\beta,\beta,2\alpha+1;y,y/(y-1)) = F\left(\left.\begin{matrix}\alpha,\beta\\ \alpha+\frac{1}{2}\end{matrix}\right|\frac{y^2}{4(y-1)}\right). \tag{94}$$

8 Remarks

The referee has kindly pointed out that formulas (11)–(9) can be proved in a more direct and simpler way. We left our original proofs in the body of the paper because they also prove generalizations of (11)–(9), as we saw in the previous sections. The referee's proof is interesting, brief, and provides an alternate explanation of the source of Gessel's formulas.

The referee's master formula is

$$\begin{aligned} F_1(\alpha,\beta,\beta,\gamma;y,z) \quad &= \quad \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k(\gamma-\alpha)_k}{k!(\gamma/2)_k((\gamma+1)/2)_k}\left(\frac{y+z}{4}\right)^k \\ &\qquad \times {}_3F_2\left(\left.\begin{matrix}\beta+k,(\alpha+k)/2,(\alpha+k+1)/2\\ k+\gamma/2,k+(\gamma+1)/2\end{matrix}\right| y+z-yz\right). \end{aligned} \tag{95}$$

Proof. From (7) it follows that

$$F_1(\alpha,\beta,\beta,\gamma;y,z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}\,y^k\,F\left(\left.\begin{matrix}-k,\ \beta\\ 1-\beta-k\end{matrix}\right|\frac{z}{y}\right).$$

Appying (29) to the right-hand side in the above identity we are led to

$$F_1(\alpha,\beta,\beta,\gamma;y,z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{m!(\gamma)_m} y^{-m}(y+z)^m F\left(\left.\begin{matrix} -m/2, (1-m)/2 \\ 1-\beta-k \end{matrix}\right| \frac{4yz}{(y+z)^2}\right).$$

Let j be the summation index in the F series on the right-hand side of the above equation. Thus $m \geq 2j$. Replace m by $m+2j$ and use the duplication formula (19), and the fact that $(\beta)_m/(1-\beta-m)_j = (-1)^j(\beta)_{m-j}$ to get (94). □

Observe that if $z = y/(y-1)$ then the argument in the ${}_3F_2$ vanishes and we obtain

$$F_1(\alpha,\beta,\beta,\gamma;y,y/(y-1)) = {}_3F_2\left(\left.\begin{matrix} \alpha,\beta,\gamma-\alpha \\ \gamma/2,(\gamma+1)/2 \end{matrix}\right| \frac{y^2}{4(y-1)}\right). \tag{96}$$

Now (95) with $\gamma = 2\alpha$ gives (9) while the case $\gamma = 2\alpha - 1$ gives (93). Next the case $\gamma = 2\alpha$ and $\beta = \alpha + 1/2$ and [8, (2.8.6)] imply (11). Furthermore (94) with $y = -z$ gives the interesting identity

$$F_1(\alpha,\beta,\beta,\gamma;y,-y) = {}_3F_2\left(\left.\begin{matrix} \beta,\alpha/2,(\alpha+1)/2 \\ \gamma/2,(\gamma+1)/2 \end{matrix}\right| y^2\right). \tag{97}$$

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