RIGHT INVERSES OF LÉVY PROCESSES AND STATIONARY STOPPED LOCAL TIMES

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ABSTRACT. If X is a symmetric Lévy process on the line, then there exists a non–decreasing, càdlàg process H such that $X(H(x)) = x$ for all $x \geq 0$ if and only if X is recurrent and has a non–trivial Gaussian component. The minimal such H is a subordinator K. The law of K is identified and shown to be the same as that of a linear time change of the inverse local time at 0 of X. When X is Brownian motion, K is just the usual ladder times process and this result extends the classical result of Lévy that the maximum process has the same law as the local time at 0. Write G_t for last point in the range of K prior to t. In a parallel with classical fluctuation theory, the process $Z := (X_t - X_{G_t})_{t\geq 0}$ is Markov with local time at 0 given by $(X_{G_t})_{t\geq 0}$. The transition kernel and excursion measure of Z are identified. A similar programme is outlined for Lévy processes on the circle. This leads to the construction of a stopping time such that the stopped local times constitute a stationary process indexed by the circle.

1. INTRODUCTION

Let $X = (X_t, \mathbb{P}^x)$ be a *Brownian motion* on the *circle* $\mathbb{T}$ thought of as the unit interval $[0, 1[$ equipped with addition $\mod 1$. Write ℓ_t^x for the *local time* of X at position $x \in \mathbb{T}$ up to time $t \geq 0$. It was shown in [Pit96] that there are stopping times T such that the $\mathbb{T}$–indexed process $(\ell_T^x)_{x\in\mathbb{T}}$ is *stationary* under $\mathbb{P}^0$ (that is, $(\ell_T^x)_{x\in\mathbb{T}}$ and $(\ell_T^{x+y})_{x\in\mathbb{T}}$ have the same distribution for all $y \in \mathbb{T}$). The discrete state–space analogue of this question for Markov chains that are equivariant under the action of a group acting on the state–space was considered in [EP97].

Motivated by a construction in [EP97], we define as follows a particular stopping time T for X with the property that $(\ell_T^x)_{x\in\mathbb{T}}$ is stationary. For $n \in \mathbb{N}$ put

$$T_0^n := \inf\{t \geq 0 : X_t = 0\}$$
$$T_{k+1}^n := \inf\left\{t \geq T_k^n : X_t = \frac{k+1}{2^n}\right\},\ k \geq 0.$$

Note that $T_{2^n}^n$ is increasing in n, and it is not hard to see that $T := \sup_n T_{2^n}^n$ is finite $\mathbb{P}^0$–a.s. (for example, if one constructs X by wrapping a linear Brownian motion around the circle, then T is dominated by the first time that the linear Brownian motion hits the level 1 after hitting the level 0). It is not hard to see that $(\ell_T^x)_{x\in\mathbb{T}}$ is indeed stationary under $\mathbb{P}^0$ (cf. the proof of Theorem 3.1 in [EP97]).

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Note further that if we define a process $(K_x^n)_{x \geq 0}$ by

$$K_x^n := T_k^n, \; \frac{k}{2^n} \leq x < \frac{k+1}{2^n},$$

then K_x^n is increasing in n for each x and the càdlàg process $K := (K_x)_{x \geq 0}$ defined by

$$K_x := \inf_{y > x} \sup_n K_y^n$$

has the property that $\mathbb{P}^0$–a.s. $X(K_x) := x \mod 1$ for all $x \geq 0$. Observe that K is a *subordinator* under $\mathbb{P}^0$. Of course, K is just an analogue of the usual ladder times process for linear Brownian motion.

Suppose now that we let X be an arbitrary *Lévy process* on $\mathbb{T}$ and ask whether the same construction leads to a finite stopping time T and a finite valued process K, which will necessarily be a subordinator under $\mathbb{P}^0$. If T is finite and X has local times, then the local time process stopped at T is stationary in the spatial variable. More generally, if T is finite then the occupation measure process of X stopped at T is stationary in the obvious sense for random measures on $\mathbb{T}$. We also note that we can apply the recipe for defining T and K to Lévy processes on $\mathbb{R}$. We will show below that if K is finite valued and H is any non–decreasing, càdlàg process such that $X(H(x)) = x$, then $K(x) \leq H(x)$, whereas no such H exist if K is not finite valued.

In the $\mathbb{R}$–valued case, the construction of T certainly leads to a finite stopping time under $\mathbb{P}^0$ when $\mathbb{P}^x\{X \text{ hits } y\} = 1$ for all $x < y$, and X has no positive jumps. We leave the relatively straightforward proof to the reader. A much more interesting case is when the Lévy measure of X assigns positive (and possibly infinite) mass to the positive half-line, so that X no longer "creeps over levels from below".

In this paper we consider *symmetric* Lévy processes on $\mathbb{T}$ or $\mathbb{R}$ and show that T is finite if and only if X has a *non–zero Gaussian component* and, in the $\mathbb{R}$–valued case, is also *recurrent*. Moreover, we identify the distribution of the subordinator K in this case.

Suppose now that X has a non–zero Gaussian component and is also recurrent in the $\mathbb{R}$–valued case. Set

$$G_t := \sup\left([0,t] \cap \{K_x : x \geq 0\}\right),$$

with the convention that $\sup \emptyset = -\infty$, and write

$$L_t := \begin{cases} X_{G_t}, & \text{if } G_t \neq -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that L_t is just the current maximum when X is linear Brownian motion started at 0. We show that $Z := (X_t - L_t)_{t \geq 0}$ is a Markov process. In the $\mathbb{R}$–valued case, $(L_t)_{t \geq 0}$ is a local time at 0 for this process, whilst in the $\mathbb{T}$–valued case this process needs to be "unwrapped" onto the line to produce a local time.

We identify the distribution of Z and find the *excursion law* in the *Itô decomposition* of Z into a point process of excursions away from 0. Of course, when X is linear Brownian motion we just recover the usual fluctation theory and classical results for excursions below the maximum, but we get new objects when dealing with processes that have jumps. At the end of the paper we investigate some of the properties of the zero set of Z.

For the remainder of the paper we will restrict attention to the $\mathbb{R}$–valued setting and leave to the reader the straightforward formulation and proof of the corresponding $\mathbb{T}$–valued results.

2. Inverses for real–valued, càdlàg functions

Definition 2.1. Given a càdlàg function $f : \mathbb{R}_+ \to \mathbb{R}$, write $\mathcal{I}(f)$ for the class of non–decreasing, càdlàg functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(g(x)) = x$ for all $x \geq 0$. Note that if $\mathcal{I}(f)$ is non–empty, then $\check{f}$ defined by $\check{f}(x) := \inf\{g(x) : g \in \mathcal{I}(f)\}$ is also in $\mathcal{I}(f)$ and $\check{f}(x) \leq g(x)$ for any $g \in \mathcal{I}(f)$.

Remark 2.2. Note that if $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous, $f(0) \leq 0$, and the range of f contains $\mathbb{R}_+$, then $\mathcal{I}(f) \neq \emptyset$ and $\check{f}(x) = \inf\{t \geq 0 : f(t) > x\}$.

Definition 2.3. Given a càdlàg function $f : \mathbb{R}_+ \to \mathbb{R}$ and $n \in \mathbb{N}$ write

$$\begin{aligned} T_0^n(f) &:= \inf\{t \geq 0 : f(t) = 0\} \\ T_{k+1}^n(f) &:= \inf\left\{t \geq T_k^n(f) : f(t) = \frac{k+1}{2^n}\right\},\ k \geq 0, \end{aligned}$$

with the usual convention that $\inf \emptyset = \infty$. Set

$$\check{f}^n(x) := T_k^n(f),\ \frac{k}{2^n} \leq x < \frac{k+1}{2^n}.$$

Note that the quantity $\check{f}^n(x)$ is non–decreasing in both n and x.

Lemma 2.4. *The set $\mathcal{I}(f)$ is non–empty if and only if $\sup_n \check{f}^n(x) < \infty$ for all $x \geq 0$, in which case $\check{f}(x) = \inf_{y>x} \sup_n \check{f}^n(y)$.*

Proof. Suppose first that $\mathcal{I}(f)$ is non-empty. It is clear that each of the times $T_k^n(f)$ is finite and that $T_k^n(f) \leq g(k/2^n)$ for any $g \in \mathcal{I}(f)$. In particular, $T_k^n(f) \leq \check{f}(k/2^n)$. Thus $\check{f}^n(x) \leq \check{f}(x)$ for all n, k. Set $\hat{f}(x) = \sup_n \check{f}^n(x) \leq \check{f}(x)$.

We claim that $\hat{f}$ is strictly increasing. If not, then there are three dyadic rationals $a < b < c$ such that $\hat{f}(a) = \hat{f}(b) = \hat{f}(c) = s$, say. By construction, $f(\check{f}^n(a)) = a$, $f(\check{f}^n(b)) = b$, and $f(\check{f}^n(c)) = c$ for all n sufficiently large. Also, we either have $\check{f}^n(a) < s$ for all n or $\check{f}^n(a) = s$ for all n sufficiently large, with similar behaviour for $\check{f}^n(b)$ and $\check{f}^n(c)$. The only way this could possibly happen would be if $\check{f}^n(a) < s$ and $\check{f}^n(b) < s$ for all n, but this would contradict the existence of a left–limit at s for f.

Now, if $\hat{f}(x)$ is not one of the countable number of points of discontinuity of f, then $f(\hat{f}(x)) = \lim_n f(\check{f}^n(x)) = x$. Because $\hat{f}$ is strictly increasing, the set of x such that $\hat{f}(x)$ is a discontinuity point of f is also countable. Therefore, for a dense set of x we have $f(\hat{f}(x)) = x$ and hence, by the right–continuity of f, $f(\bar{f}(x)) = x$ for all x, where we set $\bar{f}(x) := \inf_{y>x} \hat{f}(y)$. Note that $\bar{f}$ is càdlàg and non–decreasing, and $\bar{f}(x) \leq \inf_{y>x} \check{f}(y) = \check{f}(x)$, so that $\bar{f} = \check{f}$, as required.

The proof of the converse is similar and is left to the reader. □

Remark 2.5. It follows from the proof of Lemma 2.4 that $\check{f}$ is strictly increasing. Of course, this is also immediate from the properties that $\check{f}$ is non–decreasing and $f(\check{f}(x)) = x$.

Definition 2.6. Suppose that $\mathcal{I}(f)$ is non–empty. Put

$$\gamma_f(t) := \sup\left([0,t] \cap \{\check{f}(x) : x \geq 0\}\right), \ t \geq 0,$$

with the convention that $\sup \emptyset = -\infty$, and

$$\tilde{f}(t) := \begin{cases} f(\gamma_f(t)), & \text{if } \gamma_f(t) \neq -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.7. *Suppose that $\mathcal{I}(f)$ is non–empty.*

(i) *The function $\tilde{f}$ is continuous and non–decreasing.*
(ii) *The function $\check{f}$ is the right–continuous functional inverse of $\tilde{f}$, that is*

$$\check{f}(x) := \inf\{t \geq 0 : \tilde{f}(t) > x\}.$$

(iii) *If for fixed $s \geq 0$ we define a càdlàg function $g : \mathbb{R}_+ \to \mathbb{R}$ by*

$$g(t) := f(s+t) - \tilde{f}(s),$$

then $\mathcal{I}(g) \neq \emptyset$ and

$$\tilde{f}(s+t) = \tilde{f}(s) + \tilde{g}(t), \ t \geq 0.$$

(iv) *Let g be as in part (iii) and suppose that $s = \check{f}(x)$ for some $x \geq 0$, then*

$$\check{f}(x+y) - \check{f}(x) = \check{g}(y), \ y \geq 0.$$

Proof. (i) It is clear that $\tilde{f}$ is non–decreasing and right–continuous.

Consider the left–continuity of $\tilde{f}$ at $t > 0$. There are four cases to consider:

(a) t is not in the closure of $\{\check{f}(x) : x \geq 0\}$,
(b) $t = \check{f}(0)$,
(c) $t = \check{f}(x)$ for some $x > 0$,
(d) $t = \sup_{w<x} \check{f}(w)$ for some $x > 0$ and $\check{f}(x) > t$.

In case (a), $\tilde{f}(s) = \tilde{f}(t)$ for all $s < t$ sufficiently close to t. Case (b) is obvious. In case (c), $\lim_{s\uparrow t} \tilde{f}(s) = \lim_{w\uparrow x} f(\check{f}(w)) = \lim_{w\uparrow x} w = x = f(\check{f}(x)) = \tilde{f}(t)$. In case (d), $\lim_{s\uparrow t} \tilde{f}(s) = x$ by the argument for case (c), and we also have $\lim_{s\uparrow t} \tilde{f}(s) \leq \tilde{f}(t) \leq \tilde{f}(\check{f}(x)) = f(\check{f}(x)) = x$.
(ii) Suppose that $\check{f}(x) = t$ for some $t \geq 0$. We have $\lim_{y\downarrow x} \check{f}(y) = t$, which implies that $\tilde{f}(u) > \tilde{f}(t)$ for all $u > t$.
(iii) Observe that $g(\check{f}(x + \tilde{f}(s)) - s) = x$, $x \geq 0$, so $\mathcal{I}(g) \neq \emptyset$ and $\check{g}(x) \leq \check{f}(x + \tilde{f}(s)) - s$, $x \geq 0$. Equivalently,

$$s + \check{g}(x - \tilde{f}(s)) \leq \check{f}(x), \ x \geq \tilde{f}(s). \tag{2.1}$$

If we set

$$f^*(x) := \begin{cases} \check{f}(x), & \text{if } x < \tilde{f}(s), \\ s + \check{g}(x - \tilde{f}(s)), & \text{if } x \geq \tilde{f}(s), \end{cases}$$

then $f(f^*(x)) = x$, and, by (2.1), $f^*(x) \leq \check{f}(x)$, $x \geq 0$. Therefore $f^* = \check{f}$. The result follows readily from this equality.
(iv) This is immediate from the proof of part (iii). □

3. Existence of inverses for Lévy processes

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}^x)$ be a symmetric Lévy process on $\mathbb{R}$. That is, X is a conservative $\mathbb{R}$–valued Hunt process such that $\mathbb{P}^x\{X_t \in A\} = \mathbb{P}^0\{x + X_t \in A\}$ and $\mathbb{P}^0\{X_t \in A\} = \mathbb{P}^0\{-X_t \in A\}$ for all $x \in \mathbb{R}$, $t \geq 0$, and Borel sets A. For the convenience of the reader, we have tried to use [Ber96] as a unified reference on Lévy processes, and we refer the reader there for original bibliographic citations.

Recall that

$$\mathbb{P}^0[\exp(i\xi X_t)] = \exp\left(-t\Psi(\xi)\right),\ \xi \in \mathbb{R},$$

where

$$\Psi(\xi) = \frac{\sigma^2}{2}\xi^2 + \int_{\mathbb{R}\setminus\{0\}} (1 - \cos\xi x)\,\nu(dx)$$

for some $\sigma \geq 0$ and symmetric measure ν such that $\int (x^2 \wedge 1)\,\nu(dx) < \infty$. Note that

$$\lim_{|\xi|\to\infty} \frac{\Psi(\xi)}{\xi^2} = \frac{\sigma^2}{2}. \tag{3.1}$$

(see Proposition I.2 of [Ber96]). When $\sigma > 0$ we say that X has a *non–trivial Gaussian component.*

Notation 3.1. For $q > 0$ set

$$\kappa(q) := \left[\frac{\sigma^2}{2\pi}\int \frac{1}{q + \Psi(\xi)}\,d\xi\right]^{-1},$$

with the convention that $0^{-1} = +\infty$ and $+\infty^{-1} = 0$.

Notation 3.2. Write $S_x := \inf\{t > 0 : X_t = x\}$ for $x \in \mathbb{R}$.

Theorem 3.3. *If X is recurrent and has non–trivial Gaussian component, then $\mathcal{I}(X) \neq \emptyset$, $\mathbb{P}^y$–a.s. for all y. Otherwise, $\mathcal{I}(X) = \emptyset$, $\mathbb{P}^y$–a.s. for all y. In the former case, $K := \check{X}$ is a subordinator under $\mathbb{P}^0$ with*

$$\mathbb{P}^0[\exp(-qK_x)] = \exp\left(-x\kappa(q)\right).$$

Proof. Put $T^n_k := T^n_k(X)$ and $K^n_x := \check{X}^n(x)$, in agreement with the notation in the Introduction.

Suppose until further notice that X has a non–trivial Gaussian component and consider first what happens under $\mathbb{P}^0$.

By Lemma 2.4 we need to show that $\sup_n K^n_x$ is finite $\mathbb{P}^0$–a.s. for each $x \geq 0$ if and only if X is recurrent.

By (3.1)

$$\int \frac{1}{q + \Psi(\xi)}\,d\xi < \infty,\ q > 0. \tag{3.2}$$

By Corollary II.20 and Theorem II.19 of [Ber96], X has continuous resolvent densities $(u^q)_{q>0}$,

$$\mathbb{P}^0[\exp(-qS_x)] = u^q(x)/u^q(0), \tag{3.3}$$

and

$$u^q(x) = \frac{1}{2\pi}\int \frac{\cos\xi x}{q + \Psi(\xi)}\,d\xi. \tag{3.4}$$

We claim that

$$\lim_{x\downarrow 0}\frac{u^q(0)-u^q(x)}{x}=\frac{1}{\sigma^2}. \tag{3.5}$$

To see this, note that

$$\frac{u^q(0)-u^q(x)}{x}=\frac{1}{2\pi}\int\frac{1-\cos\xi}{x^2(q+\Psi(\xi/x))}\,d\xi.$$

By (3.1) the integrand on the right–hand side is bounded above by $c(1-\cos\xi)\xi^{-2}$ for a suitable constant c and converges to $2(1-\cos\xi)\xi^{-2}\sigma^{-2}$ as $x\downarrow 0$. The claim (3.5) now follows from dominated convergence and the observation

$$\int\frac{1-\cos\xi}{\xi^2}\,d\xi=\pi.$$

It follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned}\mathbb{P}^0[\exp(-q\sup_n K_x^n)]&=\lim_n\mathbb{P}^0[\exp(-qK_x^n)]\\&=\lim_n\mathbb{P}^0[\exp(-qS_{2^{-n}})]^{\lfloor 2^n x\rfloor}\\&=\lim_n\left[\frac{u^q(2^{-n})}{u^q(0)}\right]^{\lfloor 2^n x\rfloor}\\&=\exp\left(-x\kappa(q)\right).\end{aligned}\tag{3.6}$$

Therefore, $\sup_n K_x^n$ will be finite $\mathbb{P}^0$–a.s. if and only if the rightmost term in (3.6) converges to 1 as $q\downarrow 0$. This, however, will occur if and only if the integral in (3.2) goes to ∞, and this, in turn, is equivalent to X being recurrent when (3.2) holds (see Theorem I.17 of [Ber96]).

If $\mathcal{I}(X)$ is non–empty $\mathbb{P}^0$–a.s. and so $K=\check{X}$ is finite valued $\mathbb{P}^0$–a.s., then it is clear from Lemma 2.4 that each K_x is a stopping time for X. Moreover, it follows straightforwardly from Lemma 2.7(iv) and the Lévy property of X that K is a subordinator under $\mathbb{P}^0$ with the stated Laplace exponent.

Now consider what happens under $\mathbb{P}^y$ for general y (but still with the assumption that X has a non–trivial Gaussian component). In order that $\mathcal{I}(X)$ is non–empty $\mathbb{P}^y$–a.s. for some $y\in\mathbb{R}$ it is necessary and sufficient that $\mathcal{I}(X)$ is non–empty $\mathbb{P}^0$–a.s. and $\mathbb{P}^y\{S_0<\infty\}=1$. From what we have seen above, both of these conditions hold for all $y\in\mathbb{R}$ when X is recurrent. On the other hand, if X is transient, then we have seen that $\mathcal{I}(X)$ is empty $\mathbb{P}^0$–a.s., and this certainly implies that $\mathcal{I}(X)$ is empty $\mathbb{P}^y$–a.s. for all $y\in\mathbb{R}$.

Suppose now that X does not have a non–trivial Gaussian component. In order that $\mathbb{P}^0\{\mathcal{I}(X)\neq\emptyset\}>0$, it must certainly be the case that $\mathbb{P}^0\{S_x<\infty\}>0$ for all $x>0$, and so we can restrict attention to X with this latter property. By Exercise II.6.5 of [Ber96], $\inf\{t>0:X_t=0\}=0$, $\mathbb{P}^0$–a.s. Thus, by Theorem II.19 of [Ber96], X has continuous resolvent densities $(u^q)_{q>0}$, and (3.3) and (3.4) hold. We can then use (3.1) and the arguments above to show that for all $x>0$, $\mathbb{P}^0[\exp(-q\sup_n K_x^n)]=0$ for all $q>0$, and so $\sup_n K_x^n=\infty$, $\mathbb{P}^0$–a.s., as required. □

Remark 3.4. If X has a non–trivial Gaussian component, then X has local times. If X is also recurrent, then the inverse local time at 0 is a subordinator with Laplace exponent a multiple of κ (see Proposition V.4 of [Ber96]). That is, the inverse local time is distributed as a linear time change of K. For Brownian motion this is equivalent to Lévy's theorem that the maximum and local time at 0 processes have the same distribution.

Remark 3.5. Set $U_x := \inf\{t \geq 0 : X_t > x\}$, $x > 0$. As was remarked to us by Jean Bertoin, the condition that X has a non–trivial Gaussian component is equivalent (in our symmetric setting) to the condition that $\mathbb{P}^0\{X_{U_x} = X_{U_x-} = x\} > 0$ for some (equivalently, all) $x > 0$. That is, there is "positive probability of creeping, rather than jumping, over the level x." This latter condition is, in turn, equivalent to the condition that the ladder height process of X is a subordinator with positive drift (see Theorem VI.19 of [Ber96] and the discussion that follows it.)

Remark 3.6. It is clear from the proof of Theorem 3.3 that $\mathcal{I}(X) \neq \emptyset$, $\mathbb{P}^y$–a.s. for all $y \in \mathbb{R}$, whenever X is a recurrent Lévy process (not necessarily symmetric) with continuous resolvent densities $(u^q)_{q>0}$ such that $\rho(q) := \lim_{h\downarrow 0}(u^q(0) - u^q(h))/h$ exists and is finite, in which case $\rho(q)$ is the Laplace exponent of the subordinator K. As was pointed out to us by the referee, the existence of this limit can be established for certain non–symmetric processes. For example, if X has only positive jumps and non–trivial Gaussian component, then one can show using Exercise VII.5.2(b) of [Ber96] that the limit exists and is given by

$$\rho(q) = \frac{2}{\sigma^2 \Phi'(q)} - \Phi(q),$$

where Φ is the inverse Laplace exponent of X.

4. The "reflected" process

NOTE: From now on we suppose that X is recurrent with non–trivial Gaussian component.

Notation 4.1. As in the Introduction, put

$$G_t := \gamma_X(t),$$

$$L_t := \tilde{X}_t,$$

and

$$Z_t := X_t - L_t.$$

Recall that a point y is said to be *regular* (resp. *instantaneous*) for a Markov process $(M_t, \mathbb{Q}^x)$ if $\inf\{t > 0 : M_t = y\} = 0$ (resp. $\inf\{t > 0 : M_t \neq y\} = 0$), $\mathbb{Q}^y$–a.s.

Theorem 4.2. (i) *The process Z is a time–homogeneous, strong Markov process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.*

(ii) *The state 0 is regular and instantaneous for Z, and L is a corresponding local time.*

Proof. Given Lemma 2.7(iii), the proof of (i) is straightforward from the homogeneity and independence of the increments of X, and follows the pattern of the proof in standard fluctuation theory that a Lévy process reflected at its current maximum is strong Markov (see Proposition VI.1 of [Ber96]).

Under $\mathbb{P}^0$, the closure of the zero set of Z contains the closure of the range of a strictly increasing subordinator K. Consequently, 0 is regular for Z. Also, $\inf\{t > 0 : X_t < 0\} = 0$, $\mathbb{P}^0$–a.s. and so 0 is instantaneous for Z (for example, by symmetry and the fact that the distribution of X_t is non–atomic we have $\mathbb{P}^0\{\exists 0 \le s \le t : X_t < 0\} \ge \mathbb{P}^0\{X_t < 0\} = 1/2$, and the result follows from the Blumenthal zero–one law). Therefore, by Theorem IV.4 of [Ber96] the process Z does have a local time at 0.

Now, by parts (i) and (ii) of Lemma 2.7, $(L_t)_{t \ge 0}$ is a continuous, non–decreasing, $(\mathcal{F})_{t \ge 0}$–adapted process such that the support of the random measure dL is contained in closure of the range of K which is, in turn, contained in the closure of the zero set of Z. Moreover, if S is a $(\mathcal{F})_{t \ge 0}$ stopping time such that $Z_S = 0$, $\mathbb{P}^0$–a.s. on $\{S < \infty\}$, then it follows from Lemma 2.7(iii) and the Lévy property of X that $((Z_{S+t}, L_{S+t} - L_S))_{t \ge 0}$ is independent of $\mathcal{F}_S$ under $\mathbb{P}^0\{\cdot \mid S < \infty\}$ and has the same law as (Z, L) under $\mathbb{P}^0$. Consequently, by Proposition IV.5 of [Ber96], L is, up to a choice of normalisation, the local time of Z at 0. □

Notation 4.3. Under the assumption that X has a non–trivial Gaussian component, the distribution of X_t, $t > 0$, has a *density* p_t under $\mathbb{P}^0$ (so that the density of X_t under $\mathbb{P}^x$ is $p_t(\cdot - x)$).

Remark 4.4. Note that p_t is differentiable and $\int |p_t'(x)|\, dx \le (\pi\sigma^2 t/2)^{-1/2}$. Consequently, the joint Laplace – Fourier transform $\int_0^\infty \int \exp(-qt + izx)\, p_t'(x)\, dx\, dt$ is well–defined and is given by $-iz/[q + \Psi(z)]$.

Theorem 4.5. *Suppose that τ is a rate q exponential time that is independent of X.*

(i) *The random variables L_τ and Z_τ are independent under $\mathbb{P}^0$.*
(ii) *The random variable L_τ has an exponential distribution under $\mathbb{P}^0$ with rate $\kappa(q)$.*
(iii) *Under $\mathbb{P}^x$ the random variable Z_τ has characteristic function*

$$\mathbb{P}^x[\exp(izZ_\tau)] = \frac{q}{q + \Psi(z)} \left[\exp(izx) - iz\frac{\sigma^2}{2\pi} \int \frac{\cos \xi x}{q + \Psi(\xi)}\, d\xi\right]$$

(iv) *Under $\mathbb{P}^x$ the random variable Z_t, $t > 0$, has a density given by*

$$\frac{\mathbb{P}^x\{Z_t \in dy\}}{dy} = p_t(y - x) + \sigma^2 \int_0^t p_s(-x)\, p_{t-s}'(y)\, ds$$

Proof. Part (i) is a general result from excursion theory, as is the claim in part (ii) that L_τ is exponential (see VI.50.4 and VI.49.5 in [RW87], respectively).

In order to compute the rate of L_τ note that

$$\begin{aligned} \mathbb{P}^0[L_\tau] &= \mathbb{P}^0\left[\int_0^\infty q \exp(-qt) L_t\, dt\right] \\ &= \mathbb{P}^0\left[\int_0^\infty \exp(-qt)\, dL_t\right] \\ &= \mathbb{P}^0\left[\int_0^\infty \exp(-qK_x)\, dx\right] \\ &= \kappa(q)^{-1} \end{aligned} \tag{4.1}$$

by Theorem 3.3.

Turning to part (iii), we first consider the case when $x = 0$. We have from parts (i) and (ii) and the identity $X_\tau = Z_\tau + L_\tau$ that

$$\begin{aligned}\mathbb{P}^0[\exp(izZ_\tau)] &= \mathbb{P}^0[\exp(izX_\tau)]/\mathbb{P}^0[\exp(izL_\tau)] \\ &= \frac{q(\kappa(q) - iz)}{(q + \Psi(z))\kappa(q)}.\end{aligned} \tag{4.2}$$

Recall from the proof of Theorem 3.3 that X has continuous resolvent densities $(u^q)_{q>0}$, and we have from (3.3) that

$$\begin{aligned}\frac{q}{q+\Psi(z)}\exp(izx) &= \mathbb{P}^x[\exp(izX_\tau)] \\ &= \mathbb{P}^x[\exp(izX_\tau),\ \tau < S_0] + \mathbb{P}^x[\exp(izX_\tau),\ \tau \ge S_0] \\ &= \mathbb{P}^x[\exp(izX_\tau),\ \tau < S_0] + \mathbb{P}^x\{\tau \ge S_0\}\,\mathbb{P}^0[\exp(izX_\tau)] \\ &= \mathbb{P}^x[\exp(izX_\tau),\ \tau < S_0] + \frac{u^q(-x)}{u^q(0)}\frac{q}{q+\Psi(z)}.\end{aligned}$$

Because $(Z_t : 0 \le t < S_0)$ has the same law under $\mathbb{P}^x$ as $(X_t : 0 \le t < S_0)$, we have by similar reasoning and using (4.2) that

$$\mathbb{P}^x[\exp(izZ_\tau)] = \mathbb{P}^x[\exp(izX_\tau),\ \tau < S_0] + \frac{u^q(-x)}{u^q(0)}\frac{q(\kappa(q)-iz)}{(q+\Psi(z))\kappa(q)}.$$

Part (iii) follows upon rearranging and using the expression for u^q in (3.4).

Part (iv) follows by inverting the joint Laplace – Fourier transform implicit in part (iii). $\square$

Remark 4.6. Suppose that X is standard Brownian motion. In this case $L_t = \max\{X_s : 0 \le s \le t\} \vee 0$ under $\mathbb{P}^x$, $x \le 0$, and so, by a celebrated theorem of Lévy, Z should be distributed as Brownian motion on the negative half–line reflected at 0. Recall for any x, y that $\int_0^\infty \exp(-qt)p_t(-x)\,dt = \exp(-\sqrt{2q}|x|)/\sqrt{2q}$ and $\int_0^\infty \exp(-qt)p_t'(y)\,dt = -\mathrm{sgn}(y)\exp(-\sqrt{2q}|y|)$. Hence, for $x \le 0$,

$$\begin{aligned}\int_0^\infty \exp(-qt)\left[\int_0^t p_s(-x)p_{t-s}'(y)\,ds\right]dt &= \begin{cases} -\exp(-\sqrt{2q}|y-x|)/\sqrt{2q}, & y > 0, \\ +\exp(-\sqrt{2q}|y+x|)/\sqrt{2q}, & y < 0,\end{cases} \\ &= \begin{cases} -\int_0^\infty \exp(-qt)p_t(y-x)\,dt, & y > 0, \\ +\int_0^\infty \exp(-qt)p_t(-y-x)\,dt, & y < 0.\end{cases}\end{aligned}$$

From Theorem 4.5(iv) we thus have for $x \le 0$ that

$$\frac{\mathbb{P}^x\{Z_t \in dy\}}{dy} = \begin{cases} 0, & y > 0, \\ \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x)^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(-y-x)^2}{2t}\right), & y < 0,\end{cases}$$

as expected. Of course, for general X with jumps it is possible that $Z_t > 0$, and so it certainly not the case for such X that Z has the same distribution as $-|X|$ under $\mathbb{P}^x$, $x \le 0$.

By Theorem 4.2 and standard results of excursion theory (see Ch. IV of [Ber96] or Ch. VI of [RW87]) the paths of Z under $\mathbb{P}^0$ can be decomposed using the local time L into a Poisson point process on $\mathbb{R}_+ \times E$, where E is the space of excursion paths from 0. That is, E is the space of càdlàg path $e : \mathbb{R}_+ \to \mathbb{R}$ such that $e(t) = e(h(e)) = 0$ for all $t \ge h(e) > 0$, where $h(e) := \inf\{t > 0 : e(t) = 0 \text{ or } e(t-) = 0\}$.

This Poisson process has intenstity $\lambda \otimes n^Z$, where λ is Lebesgue measure on $\mathbb{R}_+$ and n^Z is the σ–finite Itô excursion measure on E.

If $p_t^0(x, dy)$ is the transition kernel of the stopped process $Z(t \wedge h(Z))$ (which coincides with the transition kernel of X stopped at S_0), then n^Z is given by

$$\begin{aligned} &n^Z\{e \in E : e_{t_1} \in dx_1, \dots, e_{t_k} \in dx_k, h(e) > t_1\} \\ &\quad = n_{t_1}^Z(dx_1) p_{t_2 - t_1}^0(x_1, dx_2) \cdots p_{t_k - t_{k-1}}^0(x_{k-1}, dx_k). \end{aligned}$$

for $0 < t_1 < \cdots < t_k < \infty$, where $(n_t^Z)_{t>0}$ is a certain family of measures (the *entrance law* of the excursion measure).

Similarly, the paths of X under $\mathbb{P}^0$ can be decomposed using a local time at 0 into a Poisson process of excursions from 0. The usual choice of normalisation for the local time at 0 is such that the inverse local time is a subordinator with Laplace exponent $1/u^q(0) = \sigma^2 \kappa(q)$. Denote the corresponding excursion measure by n^X. Then n^X is Markovian with transition kernel p_t^0 and entrance law that we denote $(n_t^X)_{t>0}$.

Proposition 4.7. *The family $(n_t^Z)_{t>0}$ (and hence the measure n^Z) is characterised by*

$$\int_0^\infty \int_{-\infty}^\infty \exp(-qt + izx)\, n_t^Z(dx)\, dt = \frac{\kappa(q) - iz}{q + \Psi(z)}.$$

Thus

$$n_t^Z(dx) = \frac{1}{\sigma^2} n_t^X(dx) + p_t'(x)\, dx.$$

Proof. From VI.50.3 of [RW87], Theorem 4.5 and (4.1) we have that

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \exp(-qt + izx)\, n_t^Z(dx)\, dt &= \frac{\int_0^\infty \mathbb{P}^0[\exp(-qt + izZ_t)]\, dt}{\mathbb{P}^0\left[\int_0^\infty \exp(-qt)\, dL_t\right]} \\ &= \frac{1}{q + \Psi(z)} \left[1 - iz\kappa(q)^{-1}\right] \Big/ \kappa(q)^{-1}, \end{aligned}$$

and the claim for the joint Laplace – Fourier transform follows.

As similar argument shows that

$$\int_0^\infty \int_{-\infty}^\infty \exp(-qt + izx)\, n_t^X(dx)\, dt = \frac{\sigma^2 \kappa(q)}{q + \Psi(z)},$$

and the second claim follows by inverting transforms. □

Remark 4.8. Suppose that X is standard Brownian motion. Then

$$\frac{n_t^X(dx)}{dx} = \frac{|x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right)$$

(see (VI.50.9) of [RW87]), and so

$$\frac{n_t^Z(dx)}{dx} = \begin{cases} 0, & x > 0, \\ 2\frac{|x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right), & x < 0, \end{cases}$$

which is the entrance law for Brownian motion on the negative half–line reflected at 0 (cf. Remark 4.6).

5. Properties of the zero set

Proposition 5.1. (i) *With* $\mathbb{P}^0$ *probability* 1,

$$\limsup_{t \downarrow 0} \frac{L(t)}{(2\sigma^2 t \log|\log t|)^{1/2}} = 1.$$

(ii) *The Hausdorff and packing dimensions of the set* $\{t \geq 0 : Z_t = 0\}$ *are both* $\mathbb{P}^0$*–a.s. equal to* $1/2$.

(iii) *As* $t \downarrow 0$ *the law of the random variable* $t^{-1}G_t$ *converges weakly to an arcsine distribution.*

Proof. The key to all of the claims is the consequence of (3.1) that

$$\lim_{q \to \infty} q^{-1/2}\kappa(q) = \sqrt{2}/\sigma.$$

(i) See Exercise V.4.4(b) of [Ber96].
(ii) Note that $\{t \geq 0 : Z_t = 0\}$ differs from its closure by a countable set and the same is true of the range of K. Moreover, by Theorem IV.4(iii) of [Ber96], the closure of $\{t \geq 0 : Z_t = 0\}$ coincides with the closure of the range of K. The claim follows by known results on the Hausdorff and packing dimensions of the range of a subordinator — see the discussion around (2.10) and (2.11) in [PT96].
(iii) See Theorem III.6 of [Ber96]. ☐

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