HYPERDETERMINANTAL POINT PROCESSES

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Abstract. As well as arising naturally in the study of non-intersecting random paths, random spanning trees, and eigenvalues of random matrices, determinantal point processes (sometimes also called fermionic point processes) are relatively easy to simulate and provide a quite broad class of models that exhibit repulsion between points. The fundamental ingredient used to construct a determinantal point process is a kernel giving the pairwise interactions between points: the joint distribution of any number of points then has a simple expression in terms of determinants of certain matrices defined from this kernel. In this paper we initiate the study of an analogous class of point processes that are defined in terms of a kernel giving the interaction between $2M$ points for some integer M. The role of matrices is now played by $2M$-dimensional "hypercubic" arrays, and the determinant is replaced by a suitable generalization of it to such arrays – Cayley's first hyperdeterminant. We show that some of the desirable features of determinantal point processes continue to be exhibited by this generalization.

1. Introduction

Motivated by considerations of the behavior of fermions in quantum mechanics, determinantal point processes were introduced in [Mac75]. Surveys of their properties and numerous applications may be found in [DVJ88, Sos00, Lyo03, HKPV06, ST00, ST03a, ST03b, ST04].

We consider a certain extension of this class of point processes. In order to motivate our generalization, we first consider a particular case of the determinantal point process construction. Suppose that on some measure space $(\Sigma, \mathcal{A}, \mu)$ we have a kernel $K : \Sigma^2 \to \mathbb{C}$ that defines an L dimensional projection operator for $L^2(\mu)$. That is,

- (*self-adjoint*) $K(x; y) = \bar{K}(y; x)$ for all $x, y \in \Sigma$,
- (*non-negative definite*) $\sum_{i,j=1}^{n} K(x_i; x_j) z_i \bar{z}_j \geq 0$ for all $x_1, \ldots, x_n \in \Sigma$ and $z_1, \ldots, z_n \in \mathbb{C}$,
- (*idempotent*) $\int_\Sigma K(x; y) K(y; z)\, \mu(dy) = K(x; z)$ for all $x, z \in \Sigma$,
- (*trace* $= L$) $\int_\Sigma K(x; x)\, \mu(dx) = L$.

Date: September 17, 2008.

1991 *Mathematics Subject Classification.* Primary 15A15, 60G55; Secondary 15A60, 60E05.

Key words and phrases. fermionic point process, determinant, permanent, multi-dimensional array, hypercubic array, tensor, hyperdeterminant, symmetric group, factorial moment.

SNE supported in part by NSF grant DMS-0405778.

AG supported by the Vienna Science and Technology Fund, via the project "Correlation in quantum systems".

Note that if self-adjointness and idempotence hold, then non-negative definiteness holds automatically, because in that case

$$\begin{aligned}\sum_{i,j=1}^{n} K(x_i;x_j) z_i \bar{z}_j &= \sum_{i,j=1}^{n} \int_\Sigma K(x_i;y)\bar{K}(x_j;y)\,\mu(dy) z_i \bar{z}_j \\ &= \int_\Sigma \left| \sum_{k=1}^{n} K(x_k;y) z_k \right|^2 \mu(dy) \geq 0.\end{aligned}$$

The corresponding determinantal point process can then be thought of as an exchangeable random vector with values in Σ^L. The distribution of this random vector is a probability measure that has the density

$$(x_1, \ldots, x_L) \mapsto (L!)^{-1} \det(K(x_i;x_j))_{i,j=1}^{L}$$

with respect to the measure $\mu^{\otimes L}$.

One of the most agreeable things about this construction is that for $1 \leq N \leq L$ the N-dimensional marginal distributions of the random vector have (common) density

$$(x_1, \ldots, x_N) \mapsto (L(L-1)\cdots(L-N+1))^{-1} \det(K(x_i;x_j))_{i,j=1}^{N}$$

with respect to the measure $\mu^{\otimes N}$. Consequently, the conditional distribution of the $(N+1)^{\text{st}}$ component of the Σ^L-valued random vector given the first N components can be computed explicitly (as a constant multiple of a ratio of determinants). It is thus possible to simulate the entire Σ^L-valued random vector if one is able to simulate a general Σ-valued random variable from a knowledge of its probability density function.

Various generalizations of determinantal point processes have appeared in the literature. Note that

$$\det(K(x_i;x_j))_{i,j=1}^{L} = \sum_{\sigma \in \mathfrak{S}_L} \epsilon(\sigma) \prod_{k=1}^{L} K(x_k; x_{\sigma(k)}),$$

where $\mathfrak{S}_L$ is the symmetric group of permutations of $\{1, \ldots, L\}$ and ϵ is the usual alternating character on the symmetric group (that is, the sign of a permutation). It is natural to replace ϵ by other class functions on the symmetric group (that is, by other functions that only depend on the cycle structure of a permutation and hence are constant on conjugacy classes of the symmetric group). The most obvious choice is to replace ϵ by the trivial character which always takes the value 1, thereby turning the determinant into a permanent. Permanental point processes arise in the description of bosons and are discussed in [Mac75, HKPV06, DVJ88, ST03a, ST04]. Replacing ϵ by a general irreducible character gives the immanantal point processes of [DE00], while setting $\epsilon(\sigma) = \alpha^{L-\nu(\sigma)}$ for $-1 < \alpha < 1$ and $\nu(\sigma)$ the number of cycles of σ gives the alpha-permanental processes introduced in [VJ97] and further studied in [ST03a, HKPV06].

All of these constructions have the feature that an exchangeable joint density is built up as a linear combination of products of pairwise interactions. In this paper we investigate the possibility of building up a tractable joint density as a linear combination of products of higher order interactions. In order to accomplish such a generalization, it is necessary to have higher order counterparts for both projection kernels and determinants.

Note that $K : \Sigma^2 \to \mathbb{C}$ is the kernel of an L-dimensional projection if and only if

$$K(y; z) = \sum_{\ell=1}^{L} \phi_\ell(y)\bar{\phi}_\ell(z),$$

where $\phi_1, \ldots, \phi_L$ are orthonormal in $L^2(\mu)$. One possible $(2M)^{\text{th}}$ order extension of this second order definition is to suppose that:

- the underlying space Σ is a Cartesian product $\Sigma_1 \times \cdots \times \Sigma_M$,
- the measure μ on Σ is a product measure $\mu_1 \otimes \cdots \otimes \mu_M$,
- the functions $\phi_{m\ell} : \Sigma \to \mathbb{C}$, $1 \le m \le M$, $1 \le \ell \le L$, are given by $\phi_{m\ell}(x_1, \ldots, x_M) = \psi_{m\ell}(x_m)$, where for $1 \le m \le M$ the functions $\psi_{m\ell} : \Sigma_m \to \mathbb{C}$, $1 \le \ell \le L$, belong to $L^2(\mu_m)$ and are orthonormal in $L^2(\mu_m)$,
- the kernel $K : \Sigma^{2M} \to \mathbb{C}$ is given by

$$\begin{aligned} K(y_1, \ldots, y_M; z_1, \ldots, z_M) &:= \sum_{\ell=1}^{L} \prod_{m=1}^{M} \phi_{m\ell}(y_m)\bar{\phi}_{m\ell}(z_m) \\ &= \sum_{\ell=1}^{L} \prod_{m=1}^{M} \psi_{m\ell}(y_{mm})\bar{\psi}_{m\ell}(z_{mm}). \end{aligned}$$

Note that the integral

$$\begin{aligned} &\int_\Sigma \left[\prod_{m=1}^{M} \phi_{m\ell'(m)}(x)\bar{\phi}_{m\ell''(m)}(x) \right] \mu(dx) \\ &\quad = \prod_{m=1}^{M} \int_{\Sigma_m} \psi_{m\ell'(m)}(x_m)\bar{\psi}_{m\ell''(m)}(x_m)\, \mu_m(dx_m) \end{aligned}$$

is 1 if $\ell'(m) = \ell''(m)$ for $1 \le m \le M$, and the integral is 0 otherwise. This is analogous to the orthonormality of the functions $\phi_1, \ldots, \phi_L$ appearing in the representation above of an L-dimensional projection, and when $M = 1$ we just recover that representation.

The appropriate generalization of the determinant is given by Cayley's first hyperdeterminant that was introduced in [Cay43] and which we will describe shortly. Cayley later introduced other generalizations of the determinant that he also called hyperdeterminants and are more natural from the point of view of invariant theory – see [GKZ92, GKZ94]. Early treatments of the theory related to Cayley's original definition may be found in [Pas00, Mui60, Ric18, Ric30, Old34c, Old34b, Old34a, Old36, Old40]. More recent works are [Sok60, Sok72]. We remark that Cayley's first hyperdeterminant has been useful in matroid theory [Bar95, Gly] and we also note the interesting papers [LT03, LT04] in which the calculation of Selberg and Aomoto integrals is reduced to the evaluation of hyperdeterminants of suitable multi-dimensional arrays.

Suppose that

$$\mathbb{A}(i_1, \ldots, i_M; j_1, \ldots, j_M), \quad 1 \le i_1, \ldots, i_M, j_1, \ldots, j_M \le N,$$

is a $2M$-way hypercubic matrix (that is, $\mathbb{A}$ is a a $2M$-dimensional array or tensor that is of the same length, namely N, in each direction). Suppose further that $\mathcal{K}$ is a subset of $\{1, \dots, M\}$. We define the corresponding hyperdeterminant of $\mathbb{A}$ to be

$$\begin{aligned}\mathrm{Det}_{\mathcal{K}}(\mathbb{A}) := \frac{1}{N!} \sum_{\sigma_1 \in \mathfrak{S}_N} \cdots \sum_{\sigma_M \in \mathfrak{S}_N} \sum_{\tau_1 \in \mathfrak{S}_N} \cdots \sum_{\tau_M \in \mathfrak{S}_N} \prod_{k \in \mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ \times \prod_{n=1}^{N} \mathbb{A}(\sigma_1(n), \dots, \sigma_M(n); \tau_1(n), \dots \tau_M(n)),\end{aligned}$$

where $\mathfrak{S}_N$ is the symmetric group of permutations of $\{1, \dots, N\}$ and, as above, ϵ is the alternating character. This definition is just the usual definition of the hyperdeterminant of a general hypercubic matrix with a general "signancy", except that we have imposed the restriction that the coordinate directions of the matrix are grouped in pairs, and each coordinate direction in a pair has the same signancy. When $M = 1$, so that $\mathbb{A}$ is just an $N \times N$ matrix, $\mathrm{Det}_{\mathcal{K}}(\mathbb{A})$ is either the usual determinant or the permanent, depending on whether $\mathcal{K}$ is $\{1\}$ or $\emptyset$.

Note: From now on we will assume that $\mathcal{K}$ is non-empty.

We are now ready to define a family of exchangeable probability densities. For $1 \le N \le L$, define the function $p_N : \Sigma^N \to \mathbb{C}$ by

$$p_N(x_1, \dots, x_N) := \left(\binom{L}{N}(N!)^M\right)^{-1} \mathrm{Det}_{\mathcal{K}}(\mathbb{B}),$$

where $\mathbb{B}$ is the $2M$-way hypercubic matrix of length N given by

$$\mathbb{B}(i_1, \dots, i_M; j_1, \dots, j_M) := K(x_{i_1}, \dots, x_{i_M}; x_{j_1}, \dots, x_{j_M}).$$

That is,

$$\begin{aligned}&p_N(x_1, \dots, x_N) \\ &= \left(\binom{L}{N}(N!)^{M+1}\right)^{-1} \sum_{\sigma_1 \in \mathfrak{S}_N} \cdots \sum_{\sigma_M \in \mathfrak{S}_N} \sum_{\tau_1 \in \mathfrak{S}_N} \cdots \sum_{\tau_M \in \mathfrak{S}_N} \prod_{k \in \mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ &\quad \times \prod_{n=1}^{N} K(x_{\sigma_1(n)}, \dots, x_{\sigma_M(n)}; x_{\tau_1(n)}, \dots, x_{\tau_M(n)}), \\ &= \left(\binom{L}{N}(N!)^{M+1}\right)^{-1} \sum_{\sigma_1 \in \mathfrak{S}_N} \cdots \sum_{\sigma_M \in \mathfrak{S}_N} \sum_{\tau_1 \in \mathfrak{S}_N} \cdots \sum_{\tau_M \in \mathfrak{S}_N} \prod_{k \in \mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ &\quad \times \prod_{n=1}^{N} \sum_{\ell=1}^{L} \prod_{m=1}^{M} \psi_{m\ell}(x_{\sigma_m(n)m}) \bar{\psi}_{m\ell}(x_{\tau_m(n)m}).\end{aligned}$$

We will prove the following theorem in Section 3.

Theorem 1.1. *For $1 \le N \le L$, the function p_N is the density with respect to $\mu^{\otimes N}$ of an exchangeable probability measure on Σ^N. That is, $p_N \ge 0$,*

$$\int_{\Sigma^N} p_N(x_1, \dots, x_N)\, \mu^{\otimes N}(d(x_1, \dots, x_N)) = 1,$$

and p_N is a symmetric function of its arguments.

When $\mathcal{K} = \{1, \dots, M\}$, we can be more concrete about the structure of the distribution with density p_N. Given a subset $\mathcal{L}$ of $\{1, \dots, L\}$ of cardinality N and

$1 \le m \le M$, define a kernel $K_{\mathcal{L}}^{(m)} : \Sigma_m^2 \to \mathbb{C}$ by

$$K_{\mathcal{L}}^{(m)}(u, v) = \sum_{\ell \in \mathcal{L}} \psi_{m\ell}(u) \bar{\psi}_{m\ell}(v).$$

Note that $K_{\mathcal{L}}^{(m)}$ defines an N dimensional projection operator for $L^2(\mu_m)$. As we remarked above, this implies that the function $q_{\mathcal{L}}^{(m)} : \Sigma_m^N \to \mathbb{R}$ defined by

$$q_{\mathcal{L}}^{(m)}(u_1, \ldots, u_N) := (N!)^{-1} \det K_{\mathcal{L}}^{(m)}((u_i; u_j))_{i,j=1}^N$$

is a probability density with respect to the measure $\mu_m^{\otimes N}$. Consequently,

$$(x_1, \ldots, x_N) \mapsto \prod_{m=1}^M q_{\mathcal{L}}^{(m)}(x_{1m}, \ldots, x_{Nm})$$

is a probability density with respect to the measure $\mu^{\otimes N}$. The following result is an immediate consequence of the proof of Theorem 1.1.

Corollary 1.2. *Suppose that $\mathcal{K} = \{1, \ldots, M\}$. For $1 \le N \le L$, the probability density p_N can be represented as a mixture of probability densities*

$$p_N(x_1, \ldots, x_N) = \binom{L}{N}^{-1} \sum_{\mathcal{L}} \prod_{m=1}^M q_{\mathcal{L}}^{(m)}(x_{1m}, \ldots, x_{Nm}),$$

where the sum is over all subset $\mathcal{L}$ of $\{1, \ldots, L\}$ of cardinality N.

When the underlying kernel is a projection, the marginals of a determinantal point process thought of as an exchangeable random vector are also given by a determinantal construction with the same kernel. Consequently, the marginals of a tensor product of such determinantal distributions are also given by tensor products of determinantal distributions. This observation combined with Corollary 1.2 allows one to obtain the various marginal densities of p_N when $\mathcal{K} = \{1, \ldots, M\}$.

For a general choice of $\mathcal{K}$, the function p_N is the density with respect to $\mu^{\otimes N}$ of a probability measure on $\Sigma^N = (\Sigma_1 \times \cdots \times \Sigma_M)^N \simeq \Sigma_1^N \times \cdots \times \Sigma_M^N$. We show in Section 4 that the marginal of this probability measure on $\Sigma_1^N \times \cdots \times \Sigma_{M'}^N$ for $1 \le M' < M$ is also given by a hyperdeterminantal construction (with the kernel K replaced by a suitable function of $2M'$ variables) as long as $\mathcal{K} \cap \{1, \ldots, M'\} \neq \emptyset$.

If we regard the exchangeable probability measure on Σ^N with density p_N as the distribution of a point process on Σ, then it is natural to inquire about the distribution of the number of points that fall into a given subset of Σ. We describe these distributions for the case $\mathcal{K} = \{1, \ldots, M\}$ in Section 5.

The key observation behind many of our arguments is an expansion of suitable hyperdeterminants that is analogous to the Cauchy-Binet theorem for ordinary determinants. This result is an extension of a lemma from [Bar95], and we give the proof in Section 2.

2. A hyperdeterminant expansion

For $M = 1$, the following result is a consequence of the Cauchy-Binet expansion for determinants. (Recall our assumption that $\mathcal{K}$ is non-empty and so our hyperdeterminant for $M = 1$ is a determinant rather than a permanent – the Cauchy-Binet expansion for permanents is somewhat different and involves sums over possibly repeated indices.) When $M > 1$ and $\mathcal{K} = \{1, 2, \ldots, M\}$, the result is given by Lemma 3.3 of [Bar95].

Proposition 2.1. *Suppose that $\mathbb{A}$ is a $2M$-way hypercubic matrix with length N in each direction that is of the form*

$$\mathbb{A}(i_1, \ldots, i_M; j_1, \ldots, j_M) = \sum_{\ell=1}^{L} A^{(1)}(i_1, \ell) \cdots A^{(M)}(i_M, \ell) \bar{A}^{(1)}(j_1, \ell) \cdots \bar{A}^{(M)}(j_M, \ell),$$

where $A^{(m)}$ is an $N \times L$ matrix and $\bar{A}^{(m)}$ is the $N \times L$ matrix obtained by taking the complex conjugates of the entries of $A^{(m)}$. Then $\operatorname{Det}_{\mathcal{K}}(\mathbb{A}) = 0$ if $L < N$ and otherwise

$$\begin{aligned}\operatorname{Det}_{\mathcal{K}}(\mathbb{A}) &= \sum_{\mathcal{L}} \left[\prod_{k\in\mathcal{K}} \det(A^{(k)}_{\mathcal{L}}) \det(\bar{A}^{(k)}_{\mathcal{L}})\right] \left[\prod_{k\notin\mathcal{K}} \operatorname{per}(A^{(k)}_{\mathcal{L}}) \operatorname{per}(\bar{A}^{(k)}_{\mathcal{L}})\right] \\ &= \sum_{\mathcal{L}} \left[\prod_{k\in\mathcal{K}} |\det(A^{(k)}_{\mathcal{L}})|^2\right] \left[\prod_{k\notin\mathcal{K}} |\operatorname{per}(A^{(k)}_{\mathcal{L}})|^2\right],\end{aligned}$$

where the sum is over all subsets $\mathcal{L}$ of $\{1, 2, \ldots, L\}$ with cardinality N and $A^{(k)}_{\mathcal{L}}$ is the $N \times N$ sub-matrix of the matrix $A^{(k)}$ formed by the columns of $A^{(k)}$ with indices in the set $\mathcal{L}$.

Proof. We have

$$\begin{aligned}&\operatorname{Det}_{\mathcal{K}}(\mathbb{A}) \\ &= \frac{1}{N!} \sum_{\sigma_1\in\mathfrak{S}_N} \cdots \sum_{\sigma_M\in\mathfrak{S}_N} \sum_{\tau_1\in\mathfrak{S}_N} \cdots \sum_{\tau_M\in\mathfrak{S}_N} \prod_{k\in\mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ &\quad \times \prod_{n=1}^{N} \mathbb{A}(\sigma_1(n), \ldots, \sigma_M(n); \tau_1(n), \ldots \tau_M(n)) \\ &= \frac{1}{N!} \sum_{\sigma_1\in\mathfrak{S}_N} \cdots \sum_{\sigma_M\in\mathfrak{S}_N} \sum_{\tau_1\in\mathfrak{S}_N} \cdots \sum_{\tau_M\in\mathfrak{S}_N} \prod_{k\in\mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ &\quad \times \prod_{n=1}^{N} \sum_{\ell=1}^{L} A^{(1)}(\sigma_1(n), \ell) \cdots A^{(M)}(\sigma_M(n), \ell) \bar{A}^{(1)}(\tau_1(n), \ell) \cdots \bar{A}^{(M)}(\tau_M(n), \ell) \\ &= \frac{1}{N!} \sum_{\sigma_1\in\mathfrak{S}_N} \cdots \sum_{\sigma_M\in\mathfrak{S}_N} \sum_{\tau_1\in\mathfrak{S}_N} \cdots \sum_{\tau_M\in\mathfrak{S}_N} \prod_{k\in\mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ &\quad \times \sum_{\ell_1=1}^{L} \cdots \sum_{\ell_N=1}^{L} \prod_{n=1}^{N} A^{(1)}(\sigma_1(n), \ell_n) \cdots A^{(M)}(\sigma_M(n), \ell_n) \\ &\qquad \times \bar{A}^{(1)}(\tau_1(n), \ell_n) \cdots \bar{A}^{(M)}(\tau_M(n), \ell_n) \\ &= \frac{1}{N!} \sum_{\ell_1=1}^{L} \cdots \sum_{\ell_N=1}^{L} S(\ell_1, \ell_2, \cdots, \ell_N)\ ,\end{aligned}$$

where

$$\begin{aligned}S(\ell_1, \ell_2, \cdots, \ell_N) \ &= \ \sum_{\sigma_1\in\mathfrak{S}_N} \cdots \sum_{\sigma_M\in\mathfrak{S}_N} \sum_{\tau_1\in\mathfrak{S}_N} \cdots \sum_{\tau_M\in\mathfrak{S}_N} \prod_{k\in\mathcal{K}} \epsilon(\sigma_k)\epsilon(\tau_k) \\ &\times \prod_{n=1}^{N} A^{(1)}(\sigma_1(n), \ell_n) \cdots A^{(M)}(\sigma_M(n), \ell_n) \bar{A}^{(1)}(\tau_1(n), \ell_n) \cdots \bar{A}^{(M)}(\tau_M(n), \ell_n).\end{aligned}$$

For $\vec{\ell} = (\ell_1, \dots, \ell_N) \in \{1, \dots, L\}^N$ and $m \in \{1, \dots, M\}$ define an $N \times N$ matrix $B^{(m)}_{\vec{\ell}}$ by

$$B^{(m)}_{\vec{\ell}}(i,j) := A^{(m)}(i, \ell_j).$$

Then

$$\begin{aligned} S(\vec{\ell}\,) &= \left[\prod_{k\in\mathcal{K}} \left(\sum_{\sigma\in\mathfrak{S}_N} \epsilon(\sigma) \prod_{n=1}^N A^{(k)}(\sigma(n), \ell_n)\right) \left(\sum_{\tau\in\mathfrak{S}_N} \epsilon(\tau) \prod_{n=1}^N \bar{A}^{(k)}(\tau(n), \ell_n)\right)\right] \\ &\quad \times \left[\prod_{k\notin\mathcal{K}} \left(\sum_{\sigma\in\mathfrak{S}_N} \prod_{n=1}^N A^{(k)}(\sigma(n), \ell_n)\right) \left(\sum_{\tau\in\mathfrak{S}_N} \prod_{n=1}^N \bar{A}^{(k)}(\tau(n), \ell_n)\right)\right] \\ &= \left[\prod_{k\in\mathcal{K}} \det(B^{(k)}_{\vec{\ell}}) \det(\bar{B}^{(k)}_{\vec{\ell}})\right] \left[\prod_{k\notin\mathcal{K}} \operatorname{per}(B^{(k)}_{\vec{\ell}}) \operatorname{per}(\bar{B}^{(k)}_{\vec{\ell}})\right] \\ &= \left[\prod_{k\in\mathcal{K}} |\det(B^{(k)}_{\vec{\ell}})|^2\right] \left[\prod_{k\notin\mathcal{K}} |\operatorname{per}(B^{(k)}_{\vec{\ell}})|^2\right] \end{aligned}$$

Note that the rightmost product is zero unless the entries of the vector $\vec{\ell}$ are distinct, because in that case each of the matrices $B^{(k)}_{\vec{\ell}}$ for $k \in \mathcal{K}$ will have two equal columns and hence have zero determinant (recall that $\mathcal{K}$ is non-empty). Moreover, if $\vec{\ell'} = (\ell'_1, \dots, \ell'_N)$ and $\vec{\ell''} = (\ell''_1, \dots, \ell''_N)$ are two vectors with distinct entries such that $\{\ell'_1, \dots, \ell'_N\} = \{\ell''_1, \dots, \ell''_N\} = \mathcal{L}$, then

$$|\det(B^{(k)}_{\vec{\ell}})|^2 = |\det(A^{(k)}_{\mathcal{L}})|^2$$

and

$$|\operatorname{per}(B^{(k)}_{\vec{\ell}})|^2 = |\operatorname{per}(A^{(k)}_{\mathcal{L}})|^2$$

for all k, because permuting the columns of a matrix leaves the permanent unchanged and either leaves the determinant unchanged or alters its sign.

The result now follows, because for any subset $\mathcal{L}$ of $\{1, 2, \dots, L\}$ with cardinality N there are $N!$ vectors $\vec{\ell} = (\ell_1, \dots, \ell_N)$ with $\{\ell_1, \dots, \ell_N\} = \mathcal{L}$. □

3. Proof of Theorem 1.1

By definition,

$$\binom{L}{N}(N!)^M p_N(x_1, \dots, x_N) = \operatorname{Det}_{\mathcal{K}}(\mathbb{B}),$$

where $\mathbb{B}$ is the $2M$-way hypercubic matrix of length N given by

$$\begin{aligned} \mathbb{B}(i_1, \dots, i_M; j_1, \dots, j_M) &= K(x_{i_1}, \dots, x_{i_M}; x_{j_1}, \dots, x_{j_M}) \\ &= \sum_{\ell=1}^L \prod_{m=1}^M \phi_{m\ell}(x_{i_m}) \bar{\phi}_{m\ell}(x_{j_m}) \\ &= \sum_{\ell=1}^L B^{(1)}(i_1, \ell) \cdots B^{(M)}(i_M, \ell) \bar{B}^{(1)}(j_1, \ell) \cdots \bar{B}^{(M)}(j_M, \ell), \end{aligned}$$

and the $N \times L$ matrix $B^{(m)}$ is given by

$$B^{(m)}(n, \ell) := \phi_{m\ell}(x_n) = \psi_{m\ell}(x_{nm}).$$

By Proposition 2.1,

$$\binom{L}{N}(N!)^M p_N(x_1,\ldots,x_N) = \sum_{\mathcal{L}} \left[\prod_{k\in\mathcal{K}} |\det(B_{\mathcal{L}}^{(k)})|^2\right]\left[\prod_{k\notin\mathcal{K}} |\operatorname{per}(B_{\mathcal{L}}^{(k)})|^2\right],$$

where the sum is over all subsets $\mathcal{L}$ of $\{1,2,\ldots,L\}$ with cardinality N and $B_{\mathcal{L}}^{(k)}$ is the $N\times N$ sub-matrix of the matrix $B^{(k)}$ formed by the columns of $B^{(k)}$ with indices in the set $\mathcal{L}$.

It follows that $p_N(x_1,\ldots,x_N)\geq 0$. Also, since the value of the permanent of a matrix is unchanged by a permutation of the rows and the value of a determinant is either unchanged or merely changes sign, the function p_N is unchanged by a permutation of its arguments.

We have

$$\begin{aligned}
&\binom{L}{N}(N!)^M p_N(x_1,\ldots,x_N)\\
&= \sum_{\mathcal{L}}\left[\prod_{k\in\mathcal{K}}\sum_{\sigma_k}\sum_{\tau_k}\epsilon(\sigma_k)\epsilon(\tau_k)\prod_{n=1}^N \phi_{k\sigma_k(n)}(x_n)\bar{\phi}_{k\tau_k(n)}(x_n)\right]\\
&\quad\times\left[\prod_{k\notin\mathcal{K}}\sum_{\sigma_k}\sum_{\tau_k}\prod_{n=1}^N \phi_{k\sigma_k(n)}(x_n)\bar{\phi}_{k\tau_k(n)}(x_n)\right]\\
&= \sum_{\mathcal{L}}\sum_{\sigma_1}\cdots\sum_{\sigma_M}\sum_{\tau_1}\cdots\sum_{\tau_M}\prod_{k\in\mathcal{K}}\epsilon(\sigma_k)\epsilon(\tau_k)\prod_{m=1}^M\prod_{n=1}^N \phi_{m\sigma_m(n)}(x_n)\bar{\phi}_{m\tau_m(n)}(x_n),
\end{aligned}$$

where σ_k and τ_k in the summations range over bijective maps from $\{1,\ldots,N\}$ to $\mathcal{L}$ and ϵ is interpreted in the usual way for such a bijection.

Now the integral

$$\begin{aligned}
&\int_\Sigma\left[\prod_{m=1}^M \phi_{m\sigma_m(n)}(x_n)\bar{\phi}_{m\tau_m(n)}(x_n)\right]\mu(dx_n)\\
&\quad= \prod_{m=1}^M\int_{\Sigma_m}\psi_{m\sigma_m(n)}(x_{nm})\bar{\psi}_{m\tau_m(n)}(x_{nm})\,\mu_m(dx_{nm})
\end{aligned}$$

is equal to 1 if and only if $\sigma_m(n)=\tau_m(n)$ for $1\leq m\leq M$, and otherwise the integral is 0.

Thus the integral

$$\int_{\Sigma^N}\prod_{n=1}^N\prod_{m=1}^M \phi_{m\sigma_m(n)}(x_n)\bar{\phi}_{m\tau_m(n)}(x_n)\,\mu^{\otimes N}(d(x_1,\ldots,x_N))$$

is equal to 1 if and only if $\sigma_m(n)=\tau_m(n)$ for $1\leq m\leq M$ and $1\leq n\leq N$ (equivalently, $\sigma_m=\tau_m$ for $1\leq m\leq M$), and otherwise the integral is 0.

Therefore,

$$\begin{aligned}
&\binom{L}{N}(N!)^M \int_{\Sigma^N} p_N(x_1,\dots,x_N)\,\mu^{\otimes N}(d(x_1,\dots,x_N)) \\
&= \sum_{\mathcal{L}}\sum_{\sigma_1}\cdots\sum_{\sigma_M}\prod_{k\in\mathcal{K}}[\epsilon(\sigma_k)]^2 \\
&= \#(\mathcal{L})\#(\mathfrak{S}_N)^M \\
&= \binom{L}{N}(N!)^M,
\end{aligned}$$

and

$$\int_{\Sigma^N} p_N(x_1,\dots,x_N)\,\mu^{\otimes N}(d(x_1,\dots,x_N)) = 1.$$

4. Varying the order M

Beginning with a suitable kernel $K : \Sigma^{2M} = (\Sigma_1 \times \cdots \times \Sigma_M)^{2M} \to \mathbb{C}$, we have built a family of functions p_N, $1 \le N \le L$, where p_N is a probability density on $(\Sigma_1 \times \cdots \times \Sigma_M)^N \simeq \Sigma_1^N \times \cdots \times \Sigma_M^N$ with respect to the measure $(\bigotimes_{m=1}^M \mu_m)^{\otimes N} \simeq \bigotimes_{m=1}^M \mu_m^{\otimes N}$. For $1 \le M' < M$, it is natural to ask about the push-forward of the probability measure corresponding to the density p_N by the projection map from $\bigotimes_{m=1}^M \mu_m^{\otimes N}$ to $\bigotimes_{m=1}^{M'} \mu_m^{\otimes N}$ given by

$$(x_{nm})_{1\le n\le N,\,1\le m\le M} \mapsto (x_{nm})_{1\le n\le N,\,1\le m\le M'}.$$

The answer is given by repeated applications of the following result.

Theorem 4.1. *Suppose that either $M \notin \mathcal{K}$ or $M \in \mathcal{K}$ and $\mathcal{K}\setminus\{M\} \neq \emptyset$. Set $\hat{\mathcal{K}} := \mathcal{K}\setminus\{M\}$, $\hat{\Sigma} := \prod_{i=1}^{M-1}\Sigma_m$, and $\hat{\mu} = \bigotimes_{i=1}^{M-1}\mu_m$. Define a kernel $\hat{K} : \hat{\Sigma}^{2(M-1)} \to \mathbb{C}$ by*

$$\hat{K}(y_1,\dots,y_{M-1};z_1,\dots,z_{M-1}) := \sum_{\ell=1}^{L}\prod_{m=1}^{M-1}\phi_{m\ell}(y_m)\bar{\phi}_{m\ell}(z_m).$$

The function

$$\begin{aligned}
&(x_{nm})_{1\le n\le N,\,1\le m\le M-1} \mapsto \hat{p}_N((x_{nm})_{1\le n\le N,\,1\le m\le M-1}) \\
&:= \int_{\Sigma_M^N} p_N((x_{nm})_{1\le n\le N,\,1\le m\le M})\,\mu_M^{\otimes N}(d(x_{1M},\dots,x_{NM}))
\end{aligned}$$

is a probability density with respect to the measure $\hat{\mu}^{\otimes N}$. The probability density $\hat{p}_N$ is constructed from the kernel $\hat{K}$ and the set of indices $\hat{\mathcal{K}}$ in the same manner that the probability density p_N is constructed from the kernel K and the set of indices $\mathcal{K}$.

Proof. As in the proof of Theorem 1.1,

$$\begin{aligned}
&\binom{L}{N}(N!)^M\; p_N((x_{nm})_{1\le n\le N,\,1\le m\le M}) \\
&= \sum_{\mathcal{L}}\sum_{\sigma_1}\cdots\sum_{\sigma_M}\sum_{\tau_1}\cdots\sum_{\tau_M}\prod_{k\in\mathcal{K}}\epsilon(\sigma_k)\epsilon(\tau_k)\prod_{m=1}^{M}\prod_{n=1}^{N}\psi_{m\sigma_m(n)}(x_{nm})\bar{\psi}_{m\tau_m(n)}(x_{nm}),
\end{aligned}$$

where σ_k and τ_k in the summations range over bijective maps from $\{1,\dots,N\}$ to $\mathcal{L}$.

Observe that the integral

$$\int_{\Sigma_M^N} \left[\prod_{n=1}^N \psi_{M\sigma_M(n)}(x_{nM})\bar{\psi}_{M\tau_M(n)}(x_{nM})\right] \mu_M^{\otimes N}(d(x_{1M},\ldots,x_{NM}))$$

is 1 if and only if $\sigma_M(n) = \tau_M(n)$ for $1 \le n \le N$ (that is, if and only if $\sigma_M = \tau_M$), and the integral is 0 otherwise. For each choice of the set $\mathcal{L}$, there are $N!$ choices of the pair of bijections (σ_M, τ_M) such that $\sigma_M = \tau_M$, and for all of these choices we have, of course, that $\epsilon(\sigma_M) = \epsilon(\tau_M)$ if $M \in \mathcal{K}$.

It follows that

$$\begin{aligned}
&\binom{L}{N}(N!)^M \int_{\Sigma_M^N} p_N((x_{nm})_{1\le n\le N,\, 1\le m\le M})\, \mu_M^{\otimes N}(d(x_{1M},\ldots,x_{NM})) \\
&\quad = N! \sum_{\mathcal{L}}\sum_{\sigma_1}\cdots\sum_{\sigma_{M-1}}\sum_{\tau_1}\cdots\sum_{\tau_{M-1}}\prod_{k\in\hat{\mathcal{K}}}\epsilon(\sigma_k)\epsilon(\tau_k) \\
&\qquad \times \prod_{m=1}^{M-1}\prod_{n=1}^{N}\psi_{m\sigma_m(n)}(x_{nm})\bar{\psi}_{m\tau_m(n)}(x_{nm}),
\end{aligned}$$

as claimed. □

5. The number of points falling in a set

Suppose in this section that $\mathcal{K} = \{1,\ldots,M\}$. Write $(X_1,\ldots,X_N)$ for a Σ^N-valued random variable that has the distribution possessing density p_N with respect to the measure $\mu^{\otimes N}$. Fix a set $C \in \mathcal{A}$ and let $\bar{J} := \#\{1 \le n \le N : X_n \in C\}$, so that J is a random variable taking values in the set $\{0,1,\ldots,N\}$. The distribution of the random variable $\bar{J}$ is determined by the factorial moments $\mathbb{E}[\bar{J}(\bar{J}-1)\cdots(\bar{J}-k+1)]$, $1 \le k \le N$.

Given a subset $\mathcal{L}$ of $\{1,\ldots,L\}$ of cardinality N, write $(X_1^{(\mathcal{L})},\ldots,X_N^{(\mathcal{L})})$ for a Σ^N-valued random variable that has the distribution possessing density $(x_1,\ldots,x_N) \mapsto \prod_{m=1}^M q_{\mathcal{L}}^{(m)}(x_{1m},\ldots,x_{Nm})$ with respect to the measure $\mu^{\otimes N}$. It follows from Corollary 1.2 that

$$\mathbb{E}[\bar{J}(\bar{J}-1)\cdots(\bar{J}-k+1)] = \binom{L}{N}^{-1}\sum_{\mathcal{L}}\mathbb{E}[\bar{J}^{(\mathcal{L})}(\bar{J}^{(\mathcal{L})}-1)\cdots(\bar{J}^{(\mathcal{L})}-k+1)],$$

where $\bar{J}^{(\mathcal{L})} := \#\{1 \le n \le N : X_n^{(\mathcal{L})} \in C\}$.

Note that $\bar{J}^{(\mathcal{L})} = \bar{I}_1^{(\mathcal{L})} + \cdots + \bar{I}_N^{(\mathcal{L})}$, where $\bar{I}_n^{(\mathcal{L})}$ is the indicator of the event $\{X_n^{(\mathcal{L})} \in C\}$. Suppose further that $C = C_1 \times\cdots\times C_M$, with $C_m \subseteq \Sigma_m$, $1 \le m \le M$. Then, $\bar{I}_n^{(\mathcal{L})} = \bar{I}_{n1}^{(\mathcal{L})}\cdots\bar{I}_{nM}^{(\mathcal{L})}$, where $\bar{I}_{nm}^{(\mathcal{L})}$ is the indicator of the event $\{X_{nm}^{(\mathcal{L})} \in C_m\}$.

Now, the random vector $(X_{1m}^{(\mathcal{L})},\ldots,X_{Nm}^{(\mathcal{L})})$ has density $q_{\mathcal{L}}^{(m)}$ with respect to the measure $\mu_m^{\otimes N}$, and these random vectors are independent as m varies. It follows from known results about determinantal point processes (see [ST03b, HKPV06]) that $J_m^{(\mathcal{L})} := \bar{I}_{1m}^{(\mathcal{L})} + \cdots + \bar{I}_{Nm}^{(\mathcal{L})} = \#\{1 \le n \le N : X_{nm}^{(\mathcal{L})} \in C_m\}$ has the same distribution as $\hat{I}_{1m}^{(\mathcal{L})} + \cdots + \hat{I}_{Nm}^{(\mathcal{L})}$, where the random variables $\hat{I}_{nm}^{(\mathcal{L})}$ are independent Bernoulli random variables with respective success probabilities given by the eigenvalues of the kernel $K_{\mathcal{L}}^{(m)}$ restricted to $C_m \times C_m$.

The distribution of $\bar{J} = \#\{1 \le n \le N : X_n \in C\}$ can therefore be determined using the following lemma.

Lemma 5.1. *Suppose that* $(I_{nm})_{1\le n\le N, 1\le m\le M}$ *is an array of Bernoulli random variables such that for* $1 \le m \le M$ *each of the random vectors* $(I_{1m}, \dots I_{Nm})$ *is exchangeable and the collection of random vectors* $(I_{1m}, \dots, I_{Nm})_{1\le m\le M}$ *is independent. Suppose also that* $J_m := I_{1m} + \cdots + I_{Nm}$ *has the same distribution as* $\tilde{I}_{1m} + \cdots + \tilde{I}_{Nm}$*, where* $\tilde{I}_{1m}, \dots, \tilde{I}_{Nm}$ *are independent Bernoulli random variables with* $\tilde{I}_{nm}$ *having success probability* r_{nm}*. Set* $I_n := I_{n1} \cdots I_{nM}$ *for* $1 \le n \le N$ *and* $J := I_1 + \cdots + I_N$*. Then the distribution of* J *is determined by*

$$\mathbb{E}[J(J-1)\cdots(J-k+1)] = N(N-1)\cdots(N-k+1)\binom{N}{k}^{-M}\prod_{m=1}^{M}\left(\sum_{\mathcal{S}}\prod_{n\in\mathcal{S}} r_{nm}\right),$$

where the sums are over the subsets $\mathcal{S}$ *of* $\{1, \dots, N\}$ *of cardinality* k*.*

Proof. Observe that

$$\begin{aligned}\mathbb{E}[J(J-1)\cdots(J-k+1)] &= k!\,\mathbb{E}\left[\sum_{\mathcal{S}}\prod_{n\in\mathcal{S}} I_n\right] \\ &= N(N-1)\cdots(N-k+1)\mathbb{E}[I_1\cdots I_k],\end{aligned}$$

where the sum inside the second expectation is over all subsets of $\{1, \dots, N\}$ of cardinality k, and we have used the exchangeability of $(I_1, \dots, I_N)$ for the second equality.

By the assumed independence,

$$\begin{aligned}\mathbb{E}[I_1\cdots I_k] &= \mathbb{E}[(I_{11}\cdots I_{1M})\cdots(I_{k1}\cdots I_{kM})] \\ &= \mathbb{E}[(I_{11}\cdots I_{k1})\cdots(I_{1M}\cdots I_{kM})] \\ &= \mathbb{E}[I_{11}\cdots I_{k1}]\cdots\mathbb{E}[I_{1M}\cdots I_{kM}] \\ &= \prod_{m=1}^{M}\left(\frac{1}{N(N-1)\cdots(N-k+1)}\mathbb{E}[J_m(J_m-1)\cdots(J_m-k+1)]\right) \\ &= \prod_{m=1}^{M}\left(\frac{1}{N(N-1)\cdots(N-k+1)}\,k!\,\mathbb{E}\left[\sum_{\mathcal{S}}\prod_{n\in\mathcal{S}}\tilde{I}_{nm}\right]\right),\end{aligned}$$

as required. □

Acknowledgment: The authors thank the anonymous referee for a very careful reading of the paper and several helpful comments.

References

[Bar95] Alexander I. Barvinok, *New algorithms for linear k-matroid intersection and matroid k-parity problems*, Math. Programming **69** (1995), no. 3, Ser. A, 449–470. MR MR1355700 (96j:05029)

[Cay43] A. Cayley, *On the theory of determinants*, Trans. Cambridge Philos. Soc. **8** (1843), 1–16.

[DE00] Persi Diaconis and Steven N. Evans, *Immanants and finite point processes*, J. Combin. Theory Ser. A **91** (2000), no. 1-2, 305–321, In memory of Gian-Carlo Rota. MR MR1780025 (2001m:15018)

[DVJ88] D. J. Daley and D. Vere-Jones, *An introduction to the theory of point processes*, Springer Series in Statistics, Springer-Verlag, New York, 1988. MR MR950166 (90e:60060)

[GKZ92] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Hyperdeterminants*, Adv. Math. **96** (1992), no. 2, 226–263. MR MR1196989 (94g:14023)

[GKZ94] ———, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994. MR MR1264417 (95e:14045)

[Gly] David G. Glynn, *Rota's basis conjecture and Cayley's first hyperdeterminant*, Available at `http://homepage.mac.com/dglynn/.Public/Rota2.pdf`.

[HKPV06] J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág, *Determinantal processes and independence*, Probab. Surv. **3** (2006), 206–229 (electronic). MR MR2216966

[LT03] Jean-Gabriel Luque and Jean-Yves Thibon, *Hankel hyperdeterminants and Selberg integrals*, J. Phys. A **36** (2003), no. 19, 5267–5292. MR MR1985318 (2004d:15011)

[LT04] ———, *Hyperdeterminantal calculations of Selberg's and Aomoto's integrals*, Molecular Physics **102** (2004), no. 11-12, 1351–1359.

[Lyo03] Russell Lyons, *Determinantal probability measures*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 167–212. MR MR2031202 (2005b:60024)

[Mac75] Odile Macchi, *The coincidence approach to stochastic point processes*, Advances in Appl. Probability **7** (1975), 83–122. MR MR0380979 (52 #1876)

[Mui60] Thomas Muir, *A treatise on the theory of determinants*, Revised and enlarged by William H. Metzler, Dover Publications Inc., New York, 1960. MR MR0114826 (22 #5644)

[Old34a] Rufus Oldenburger, *Composition and rank of n-way matrices and multilinear forms*, Ann. of Math. (2) **35** (1934), no. 3, 622–653. MR MR1503183

[Old34b] ———, *Composition and rank of n-way matrices and multilinear forms—supplement*, Ann. of Math. (2) **35** (1934), no. 3, 654–657. MR MR1503184

[Old34c] ———, *Transposition of Indices in Multiple-Labeled Determinants*, Amer. Math. Monthly **41** (1934), no. 6, 350–356. MR MR1523115

[Old36] ———, *Non-singular multilinear forms and certain p-way matrix factorizations*, Trans. Amer. Math. Soc. **39** (1936), no. 3, 422–455. MR MR1501856

[Old40] ———, *Higher dimensional determinants*, Amer. Math. Monthly **47** (1940), 25–33. MR MR0001195 (1,194e)

[Pas00] E. Pascal, *Die Determinanten*, Teubner-Verlag, Leipzig, 1900.

[Ric18] Lepine Hall Rice, *P-Way Determinants, with an Application to Transvectants*, Amer. J. Math. **40** (1918), no. 3, 242–262. MR MR1506358

[Ric30] ———, *Introduction to higher determinants*, Journal of Mathematics and Physics (Massachusetts Institute of Technology) **9** (1930), 47–70.

[Sok60] N. P. Sokolov, *Prostranstvennye matritsy i ikh prilozheniya*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. MR MR0130256 (24 #A122)

[Sok72] ———, *Vvedenie v teoriyu mnogomernykh matrits*, Izdat. "Naukova Dumka", Kiev, 1972. MR MR0352115 (50 #4602)

[Sos00] A. Soshnikov, *Determinantal random point fields*, Uspekhi Mat. Nauk **55** (2000), no. 5(335), 107–160. MR MR1799012 (2002f:60097)

[ST00] Tomoyuki Shirai and Yoichiro Takahashi, *Fermion process and Fredholm determinant*, Proceedings of the Second ISAAC Congress, Vol. 1 (Fukuoka, 1999) (Dordrecht), Int. Soc. Anal. Appl. Comput., vol. 7, Kluwer Acad. Publ., 2000, pp. 15–23. MR MR1940779 (2004f:28007)

[ST03a] ———, *Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes*, J. Funct. Anal. **205** (2003), no. 2, 414–463. MR MR2018415 (2004m:60104)

[ST03b] ———, *Random point fields associated with certain Fredholm determinants. II. Fermion shifts and their ergodic and Gibbs properties*, Ann. Probab. **31** (2003), no. 3, 1533–1564. MR MR1989442 (2004k:60146)

[ST04] ———, *Random point fields associated with fermion, boson and other statistics*, Stochastic analysis on large scale interacting systems, Adv. Stud. Pure Math., vol. 39, Math. Soc. Japan, Tokyo, 2004, pp. 345–354. MR MR2073340

[VJ97] D. Vere-Jones, *Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions*, New Zealand J. Math. **26** (1997), no. 1, 125–149. MR MR1450811 (98j:15007)

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