THE SEMIGROUP OF COMPACT METRIC MEASURE SPACES AND ITS INFINITELY DIVISIBLE PROBABILITY MEASURES

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Abstract. A compact metric measure space is a compact metric space equipped with probability measure that has full support. Two such spaces are equivalent if they are isometric as metric spaces via an isometry that maps the probability measure on the first space to the probability measure on the second. The resulting set of equivalence classes can be metrized with the Gromov-Prohorov metric of Greven, Pfaffelhuber and Winter. We consider the natural binary operation $\boxplus$ on this space that takes two compact metric measure spaces and forms their Cartesian product equipped with the sum of the two metrics and the product of the two probability measures. We show that the compact metric measure spaces equipped with this operation form a cancellative, commutative, Polish semigroup with a translation invariant metric and that each element has a unique factorization into prime elements. Moreover, there is an explicit family of continuous semicharacters that are extremely useful in understanding the properties of this semigroup.

We investigate the interaction between the semigroup structure and the natural action of the positive real numbers on this space that arises from scaling the metric. For example, we show that for any given positive real numbers a, b, c the trivial space is the only space $\mathcal{X}$ that satisfies $a\mathcal{X} \boxplus b\mathcal{X} = c\mathcal{X}$.

We establish that there is no analogue of the law of large numbers: if $\mathbf{X}_1, \mathbf{X}_2, \dots$ is an identically distributed independent sequence of random spaces, then no subsequence of $\frac{1}{n} \boxplus_{k=1}^{n} \mathbf{X}_k$ converges in distribution unless each $\mathbf{X}_k$ is almost surely equal to the trivial space. We characterize the infinitely divisible probability measures and the Lévy processes on this semigroup, characterize the stable probability measures and establish a counterpart of the LePage representation for the latter class.

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1. Introduction

The Cartesian product $G \square H$ of two finite graphs G and H with respective vertex sets $V(G)$ and $V(H)$ and respective edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G \square H) := V(G) \times V(H)$ and edge set

$$\begin{aligned} E(G \square H) := \{((g', h), (g'', h)) : (g', g'') \in E(G),\ h \in V(H)\} \\ \cup \{((g, h'), (g, h'')) : g \in V(G),\ (h', h'') \in E(H)\}. \end{aligned}$$

This construction plays a role in many areas of graph theory. For example, it is shown in [Sab60] that any connected finite graph is isomorphic to a Cartesian product of graphs that are irreducible in the sense that they cannot be represented as Cartesian products and that this representation is unique up to the order of the factors (see, also, [Viz63, Mil70, Imr71, Wal87, AFDF00, Tar92]). The study of the problem of embedding a graph in a Cartesian product was initiated in [GW85, GW84]. A comprehensive review of factorization and embedding problems is [Win87].

If two connected finite graphs G and H are equipped with the usual shortest path metrics r_G and r_H, then the shortest path metric on the Cartesian product is given by $r_{G \times H} = r_G \oplus r_H$, where

$$\begin{aligned} (r_G \oplus r_H)((g', h'), (g'', h'')) := r_G(g', g'') + r_H(h', h''), \\ (g', h'),\ (g'', h'') \in G \times H. \end{aligned}$$

We use the notation $\oplus$ because if we think of the shortest path metric on a finite graph as a matrix, then the matrix for the shortest path metric on the Cartesian product of two graphs is the Kronecker sum of the matrices for the two graphs and the $\oplus$ notation is commonly used for the Kronecker sum [SH11].

It is natural to consider the obvious generalization of this construction to arbitrary metric spaces and there is a substantial literature in this direction. For example, a related binary operation on metric spaces is considered by Ulam [Mau81, Problem 77(b)] who constructs a metric on the Cartesian product of two metric spaces (Y, r_Y) and (Z, r_Z) via $((y', z'), (y'', z'')) \mapsto \sqrt{r_Y(y', y'')^2 + r_Z(z', z'')^2}$ and asks whether it is possible that there could be two nonisometric metric spaces U and V such that the metrics spaces $U \times U$ and $V \times V$ are isometric. An example of two such spaces is given in [Fou71]. However, it follows from the results of [Gru70, Mos92] that such an example is not possible if U and V are compact subsets of Euclidean space.

On the other hand, a classical result of de Rahm [dR52] says that a complete, simply connected, Riemannian manifold has a product decomposition $M_0 \times M_1 \times \cdots \times M_k$, where the manifold M_0 is a Euclidean space (perhaps just a point) and M_i, $i = 1, \ldots, k$, are irreducible Riemannian manifolds that each have more than one point and are not isometric to the real line. By convention, the metric on a product of manifolds is the one appearing in Ulam's problem. This last result was extended to the setting of geodesic metric spaces of finite dimension in [FL08].

Ulam's problem is closely related to the question of *cancellativity* for this binary operation; that is, if $Y \times Z'$ and $Y \times Z''$ are isometric, then are Z' and Z'' isometric? This property clearly does not hold in general; for example, $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ and $\ell^2(\mathbb{N})$ (where $\mathbb{N} := \{0, 1, 2, \ldots\}$) are isometric, but $\ell^2(\mathbb{N})$ and the trivial metric space are not isometric. Moreover, an example is given in [Her94] showing that it does not even hold for arbitrary subsets of $\mathbb{R}$. However, we note from [BP95] that there are many compact Hausdorff topological spaces K with the property that if L' and L'' are two compact Hausdorff spaces such that $K \times L'$ and $K \times L''$ are homeomorphic, then L' and L'' are homeomorphic (see also [Zer01]).

Returning to the binary operation that combines two metric spaces (Y, r_Y) and (Z, r_Z) into the metric space $(Y \times Z, r_Y \oplus r_Z)$, it is shown in [Tar92] that if a compact metric space is isometric to a product of finitely many irreducible compact metric spaces, then this factorization is unique up to the order of the factors. However, there are certainly compact metric spaces that are not isometric to a finite product of finitely many irreducible compact metric spaces and the study of this binary operation seems to be generally rather difficult.

In this paper we consider a closely-related binary operation on the class of *compact metric measure spaces*; that is, objects that consist of a compact metric space (X, r_X) equipped with a probability measure μ_X that has full support. Following [Gro99] (see, also, [Ver98, Ver03, Ver04]), we regard two such spaces as being equivalent if they are isometric as metric spaces with an isometry that maps the probability measure on the first space to the probability measure on the second. Denote by $\mathbb{K}$ the set of such equivalence classes. With a slight abuse of notation, we will not distinguish between an equivalence class $\mathcal{X} \in \mathbb{K}$ and a representative triple (X, r_X, μ_X).

Gromov and Vershik show that a compact metric measure space (X, r_X, μ_X) is uniquely determined by the distribution of the infinite random matrix of distances

$$(r_X(\xi_i, \xi_j))_{(i,j)\in\mathbb{N}\times\mathbb{N}},$$

where $(\xi_k)_{k\in\mathbb{N}}$ is an i.i.d. sample of points in X with common distribution μ_X, and this concise condition for equivalence makes metric measure spaces considerably easier to study than metric spaces *per se*.

We define a binary, associative, commutative operation $\boxplus$ on $\mathbb{K}$ as follows. Given two elements $\mathcal{Y} = (Y, r_Y, \mu_Y)$ and $\mathcal{Z} = (Z, r_Z, \mu_Z)$ of $\mathbb{K}$, let $\mathcal{Y} \boxplus \mathcal{Z}$ be $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{K}$, where

- $X := Y \times Z$,
- $r_X := r_Y \oplus r_Z$, where $(r_Y \oplus r_Z)((y', z'), (y'', z'')) = r_Y(y', y'') + r_Z(z', z'')$ for $(y', z'), (y'', z'') \in Y \times Z)$,
- $\mu_X := \mu_Y \otimes \mu_Z$.

The distribution of the random matrix of distances for $\mathcal{Y}\boxplus\mathcal{Z}$ is the convolution of the distributions of the random matrices of distances for $\mathcal{Y}$ and $\mathcal{Z}$. The equivalence class of compact metric measure spaces $\mathcal{E}$ that each consist of a single point with the only possible metric and probability measure on them is the neutral element for this operation, and so $(\mathbb{K}, \boxplus)$ is a commutative semigroup with an identity. A semigroup with an identity is sometimes called a *monoid*.

Remark 1.1. We could have chosen other ways to combine the metrics r_Y and r_Z to give a metric on $Y \times Z$ that induces the product topology and results in a counterpart of $\boxplus$ that is commutative and associative. For example, by analogy with Ulam's construction we could have used one of the metrics $((y', z'), (y'', z'')) \mapsto (r_Y(y', y'')^p + r_Z(z', z'')^p)^{\frac{1}{p}}$ for $p > 1$ or the metric $((y', z'), (y'', z'')) \mapsto r_Y(y', y'') \vee r_Z(z', z'')$. We do not investigate these possibilities here.

We finish this introduction with an overview of the remainder of the paper.

We show in Section 2 that if we equip $\mathbb{K}$ with the Gromov-Prohorov metric d_{GPr} introduced in [GPW09] (see Section 12 for the definition of d_{GPr}), then the binary operation $\boxplus : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is continuous and the metric d_{GPr} is translation invariant for the operation $\boxplus$. We recall from [GPW09] that $(\mathbb{K}, d_{\mathrm{GPr}})$ is a complete, separable metric space. Moreover, the Gromov-Prohorov metric has the property that a sequence of elements of $\mathbb{K}$ converges to an element of $\mathbb{K}$ if and only if the corresponding sequence of associated random distance matrices described above converges in distribution to the random distance matrix associated with the limit. In Section 2 we also introduce a partial order $\leqslant$ on $\mathbb{K}$ by declaring that $\mathcal{Y} \leqslant \mathcal{Z}$ if $\mathcal{Z} = \mathcal{Y} \boxplus \mathcal{X}$ for some $\mathcal{X} \in \mathbb{K}$ and show for any $\mathcal{Z} \in \mathbb{K}$ that the set $\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is compact.

A semicharacter is a map $\chi : \mathbb{K} \to [0,1]$ such that $\chi(\mathcal{Y} \boxplus \mathcal{Z}) = \chi(\mathcal{Y})\chi(\mathcal{Z})$ for all $\mathcal{Y}, \mathcal{Z} \in \mathbb{K}$. We introduce a natural family of semicharacters in Section 3. This family has the property that $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$ for some sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ and element $\mathcal{X}$ in $\mathbb{K}$ if and only if $\lim_{n\to\infty} \chi(\mathcal{X}_n) = \chi(\mathcal{X})$ for all semicharacters χ in the family. Using the semicharacters, we show that if $\lim_{n\to\infty} \boxplus_{k=0}^{n} \mathcal{X}_k$ exists for some sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$, then $\boxplus_{k=0}^{n} \mathcal{X}'_k$ converges to the same limit for any rearrangement $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ of the sequence. We also use the semicharacters in Section 4 to prove that $(\mathbb{K}, \boxplus)$ is cancellative.

We establish in Section 5 that any element of $\mathbb{K}\backslash\{\mathcal{E}\}$ has a unique representation as either a finite or countable $\boxplus$ combination of irreducible elements, and this representation is unique up to the order of the "factors". We also find that the irreducible elements are a dense, G_δ subset of $\mathbb{K}$. The unique factorization result has several consequences. For example, it follows readily from it that $\Phi : \mathbb{R}_+ \to \mathbb{K}$ is a function such that $\Phi(0) = \mathcal{E}$ and

$$\Phi(s+t) = \Phi(s) \boxplus \Phi(t), \quad 0 \leqslant s, t < \infty,$$

then $\Phi \equiv \mathcal{E}$.

In Section 6 we investigate the measure that is obtained by taking an element of $\mathbb{K}$ and assigning a unit mass to each irreducible element (counted according to multiplicity) in its factorization. We show that this mapping from elements of $\mathbb{K}$ to measures on $\mathbb{K}$ concentrated on the set of irreducible elements is measurable in a natural sense.

Given $\mathcal{X} \in \mathbb{K}$ and $a > 0$, we define the rescaled compact metric measure space $a\mathcal{X} := (X, ar_X, \mu_X) \in \mathbb{K}$. We show in Section 7 that if $(a\mathcal{X}) \boxplus (b\mathcal{X}) = c\mathcal{X}$ for some $\mathcal{X} \in \mathbb{K}$ and $a, b, c > 0$, then $\mathcal{X} = \mathcal{E}$, so the second distributivity law certainly does not hold for this scaling operation.

We begin the study of random elements of $\mathbb{K}$ in Section 8 by defining a counterpart of the usual Laplace transform in which exponential functions are replaced by semicharacters. Two random elements of $\mathbb{K}$ have the same distribution if and only if their Laplace transforms are equal.

We introduce the appropriate notion of infinitely divisible random elements of $\mathbb{K}$ in Section 9 and obtain an analogue of the classical Lévy-Hiňcin-Itô description of infinitely divisible real-valued random variables. Our approach to this result is probabilistic and involves constructing for any infinitely divisible random element a Lévy process that at time 1 has the same distribution as the given random element. Our setting resembles that of nonnegative infinitely divisible random variables and so there is no counterpart of a Gaussian component in this description. Also, there is no deterministic component: the only constant that is infinitely divisible is the trivial space $\mathcal{E}$.

Using the scaling operation on $\mathbb{K}$ we define stable random elements of $\mathbb{K}$ in Section 10. We determine how the Lévy-Hinčin-Itô description specializes to such random elements and also verify that there is a counterpart of the LePage series that represents a stable random element as an "infinite weighted sum" of independent identically distributed random elements with a suitable independent sequence of coefficients.

Lastly, for ease of reference we summarize some facts about the Gromov-Prohorov metric in Section 12.

2. Topological and order properties

Lemma 2.1. *The operation* $\boxplus : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ *is continuous. More specifically, if* $\mathcal{X}_i, \mathcal{Y}_i$, $i = 1, 2$, *are elements of* $\mathbb{K}$, *then*

$$d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{X}_2, \mathcal{Y}_1 \boxplus \mathcal{Y}_2) \leqslant d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{Y}_1) + d_{\mathrm{GPr}}(\mathcal{X}_2, \mathcal{Y}_2)\,.$$

Proof. Let ϕ_{X_i} and ϕ_{Y_i} be isometries from X_i and Y_i to a common metric measure space $\mathcal{Z}_i$, $i = 1, 2$. The combined function (ϕ_{X_1}, ϕ_{X_2}) (resp. (ϕ_{Y_1}, ϕ_{Y_2})) maps $X_1 \times X_2$ (resp. $Y_1 \times Y_2$) isometrically into $Z_1 \times Z_2$. The result now follows from Lemma 12.1. □

A proof similar to that of Lemma 2.1 using Lemma 12.2 establishes the following result.

Lemma 2.2. *The metric d_{GPr} is translation invariant for the operation* $\boxplus$. *That is, if* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$ *are elements of* $\mathbb{K}$, *then*

$$d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{Y}, \mathcal{X}_2 \boxplus \mathcal{Y}) = d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{X}_2)\,.$$

Definition 2.3. Given $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{K}$, write $\operatorname{diam}(\mathcal{X})$ for the diameter of the compact metric space X; that is,

$$\operatorname{diam}(\mathcal{X}) := \sup\{r_X(x', x'') : x', x'' \in X\}.$$

The following is obvious.

Lemma 2.4. a) *The diameter is an additive functional on* $(\mathbb{K}, \boxplus)$; *that is,*

$$\operatorname{diam}(\mathcal{X} \boxplus \mathcal{Y}) = \operatorname{diam}(\mathcal{X}) + \operatorname{diam}(\mathcal{Y})$$

for all $\mathcal{X}, \mathcal{Y} \in \mathbb{K}$.

b) *The inequality*

$$d_{\mathrm{GPr}}(\mathcal{X} \boxplus \mathcal{Y}, \mathcal{X}) \leqslant d_{\mathrm{GPr}}(\mathcal{Y}, \mathcal{E}) \leqslant \operatorname{diam}(\mathcal{Y})$$

holds for all $\mathcal{X}, \mathcal{Y} \in \mathbb{K}$.

Remark 2.5. The function diam is not continuous on $\mathbb{K}$. For example, let $\mathcal{X}_n = (\{0,1\}, r, \mu_n)$, where $r(0,1) = 1$, $\mu_n\{0\} = 1 - \frac{1}{n}$ and $\mu_n\{1\} = \frac{1}{n}$. Then, $\mathcal{X}_n$ converges to the trivial space $\mathcal{E}$, whereas $\mathrm{diam}(\mathcal{X}_n) = 1 \nrightarrow 0 = \mathrm{diam}(\mathcal{E})$. However, the function diam is lower semicontinuous; that is, if the sequence $\mathcal{X}_n$ converges to $\mathcal{X}$ in $\mathbb{K}$ as $n \to \infty$, then $\mathrm{diam}(\mathcal{X}) \leqslant \liminf_{n\to\infty} \mathrm{diam}(\mathcal{X}_n)$. To see this, suppose that the sequence $\mathcal{X}_n$ converges to $\mathcal{X}$ $(\xi_k^{(n)})_{k\in\mathbb{N}}$ are i.i.d. in X_n with the common distribution μ_{X_n}, and $(\xi_k)_{k\in\mathbb{N}}$ are i.i.d. in X with the common distribution μ_X. Observe for any k that $\max_{1\leqslant i<j\leqslant k}(r_{X_n}(\xi_i^{(n)}, \xi_j^{(n)})$ converges in distribution to $\max_{1\leqslant i<j\leqslant k}(r_X(\xi_i, \xi_j))$. It suffices to note that $\max_{1\leqslant i<j\leqslant k}(r_{X_n}(\xi_i^{(n)}, \xi_j^{(n)}))$ is increasing in k and converges almost surely to $\mathrm{diam}(\mathcal{X}_n)$ as $k \to \infty$ and that $\max_{1\leqslant i<j\leqslant k}(r_X(\xi_i, \xi_j))$ is increasing in k and converges almost surely to $\mathrm{diam}(\mathcal{X})$ as $k \to \infty$.

Definition 2.6. Define a partial order $\leqslant$ on $\mathbb{K}$ by setting $\mathcal{Y} \leqslant \mathcal{Z}$ if $\mathcal{Z} = \mathcal{Y} \boxplus \mathcal{X}$ for some $\mathcal{X} \in \mathbb{K}$.

The symmetry and transitivity of $\leqslant$ is obvious. The antisymmetry is apparent from part (a) of Lemma 2.4. This partial order is the dual of the *Green* or *divisibility* order (see [Gri01, Section I.4.1]). The identity $\mathcal{E}$ is the unique minimal element.

Lemma 2.7. a) *For any compact set $\mathbb{S} \subset \mathbb{K}$, the set $\bigcup_{\mathcal{Z}\in\mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is compact.*

b) *For any compact set $\mathbb{S} \subset \mathbb{K}$, the set $\{(\mathcal{Y}, \mathcal{Z}) \in \mathbb{K}^2 : \mathcal{Z} \in \mathbb{S}, \mathcal{Y} \leqslant \mathcal{Z}\}$ is compact.*

c) *The map K from $\mathbb{K}$ to the compact subsets of $\mathbb{K}$ defined by $K(\mathcal{X}) := \{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{X}\}$ is upper semicontinuous. That is, if $F \subset \mathbb{K}$ is closed, then $\{\mathcal{X} \in \mathbb{K} : K(\mathcal{X}) \cap F \neq \emptyset\}$ is closed. Equivalently, if $\mathcal{X}_n \to \mathcal{X}$, and $\mathcal{Y}_n \in K(\mathcal{X}_n)$ converges to $\mathcal{Y}$, then $\mathcal{Y} \in K(\mathcal{X})$.*

Proof. We first show that $\bigcup_{\mathcal{Z}\in\mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is pre-compact. Given $\varepsilon > 0$, we know from from [GPW09, Theorem 2] that there exist $K > 0$ and $\delta > 0$ such that for all $\mathcal{Z} \in \mathbb{S}$

$$\mu_Z \otimes \mu_Z\{(z', z'') \in Z \times Z : r_Z(z', z'') > K\} \leqslant \varepsilon$$

and

$$\mu_Z\{z' \in Z : \mu_Z\{z'' \in Z : r_Z(z', z'') < \varepsilon\} \leqslant \delta\} \leqslant \varepsilon.$$

If $\mathcal{Y} \leqslant \mathcal{Z}$ for some $\mathcal{Z} \in \mathbb{S}$, then, by definition, there is a $\mathcal{W} \in \mathbb{K}$ such that $\mathcal{Z} = \mathcal{Y} \boxplus \mathcal{W}$, and so

$$\begin{aligned}
&\mu_Y \otimes \mu_Y\{(y', y'') \in Y \times Y : r_Y(y', y'') > K\} \\
&\quad \leqslant (\mu_Y \otimes \mu_Y) \otimes (\mu_W \otimes \mu_W)\{((y', y''), (w', w'')) \in (Y \times Y) \times (W \times W) : \\
&\qquad\qquad r_Y(y', y'') + r_W(w', w'') > K\} \\
&\quad = \mu_Z \otimes \mu_Z\{(z', z'') \in Z \times Z : r_Z(z', z'') > K\} \\
&\quad \leqslant \varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mu_Y\{y' \in Y : \mu_Y\{y'' \in Y : r_Y(y', y'') < \varepsilon\} \leqslant \delta\} \\
&\quad = \mu_Y \otimes \mu_W\{(y'', w'') \in Y \times W : r_Y(y', y'') < \varepsilon\} \leqslant \delta\} \\
&\quad \leqslant \mu_Y \otimes \mu_W\{(y', w') \in Y \times W : \mu_Y \otimes \mu_W\{(y'', w'') \in Y \times W : \\
&\qquad\qquad r_Y(y', y'') + r_W(w', w'') < \varepsilon\} \leqslant \delta\} \\
&\quad = \mu_Z\{z' \in Z : \mu_Z\{z'' \in Z : r_Z(z', z'') < \varepsilon\} \leqslant \delta\} \\
&\quad \leqslant \varepsilon\,.
\end{aligned}$$

It follows from [GPW09, Theorem 2] that $\bigcup_{\mathcal{Z} \in \mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is pre-compact.

We now show that $\bigcup_{\mathcal{Z} \in \mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is closed, and hence compact. Suppose now that $(\mathcal{Y}_n)_{n \in \mathbb{N}}$ is a sequence in $\bigcup_{\mathcal{Z} \in \mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ that converges to a limit $\mathcal{Y}_\infty$. For each $n \in \mathbb{N}$ we can find $\mathcal{Z}_n \in \mathbb{S}$ and $\mathcal{W}_n \in \bigcup_{\mathcal{Z} \in \mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ such that $\mathcal{Z}_n = \mathcal{Y}_n \boxplus \mathcal{W}_n$. From the above we can find a subsequence $(n(k))_{k \in \mathbb{N}}$, $\mathcal{Z}_\infty \in \mathbb{S}$ and $\mathcal{W}_\infty \in \mathbb{K}$ such that $\lim_{k \to \infty} \mathcal{Z}_{n(k)} = \mathcal{Z}_\infty$ and $\lim_{k \to \infty} \mathcal{W}_{n(k)} = \mathcal{W}_\infty$. By the continuity of the semigroup operation established in Lemma 2.1,

$$\mathcal{Y}_\infty \boxplus \mathcal{W}_\infty = \lim_{k \to \infty}(\mathcal{Y}_{n(k)} \boxplus \mathcal{W}_{n(k)}) = \lim_{k \to \infty} \mathcal{Z}_{n(k)} = \mathcal{Z}_\infty,$$

which implies that $\mathcal{Y}_\infty \leqslant \mathcal{Z}_\infty \in \mathbb{S}$ (and also $\mathcal{W}_\infty \leqslant \mathcal{Z}_\infty \in \mathbb{S}$). Therefore, $\bigcup_{\mathcal{Z} \in \mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is closed and hence compact.

(b) Because $\{(\mathcal{Y}, \mathcal{Z}) \in \mathbb{K}^2 : \mathcal{Z} \in \mathbb{S}, \mathcal{Y} \leqslant \mathcal{Z}\}$ is a subset of the compact set $(\bigcup_{\mathcal{Z} \in \mathbb{S}}\{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{Z}\}) \times \mathbb{S}$, it suffices to show that the former set is closed, but this follows from an argument similar to that which completed the proof of part (a).

(c) This is immediate from (b). □

Remark 2.8. Any partially ordered space can be endowed with a corresponding Scott topology generated by the order, see [GHK+03]. In particular, the Scott topology on $(\mathbb{K}, \leqslant)$ is much weaker than the one induced by the Gromov-Prohorov metric.

3. Semicharacters

Following the standard terminology in semigroup theory, a *semicharacter* is a map $\chi : \mathbb{K} \to [0,1]$ such that $\chi(\mathcal{Y} \boxplus \mathcal{Z}) = \chi(\mathcal{Y})\chi(\mathcal{Z})$ for all $\mathcal{Y}, \mathcal{Z} \in \mathbb{K}$.

Definition 3.1. Denote by $\mathbb{A}$ the consisting of the empty set and the arrays $A = (a_{ij})_{1 \leqslant i < j \leqslant n} \in \mathbb{R}_+^{\binom{n}{2}}$ for $n \geqslant 2$. For each $A \in \mathbb{A}$ define a semicharacter χ_A by setting $\chi_\varnothing \equiv 1$ and

$$\chi_A((X, r_X, \mu_X)) := \int_{X^n} \exp\left(-\sum_{1 \leqslant i < j \leqslant n} a_{ij} r_X(x_i, x_j)\right) \mu_X^{\otimes n}(dx) \tag{3.1}$$

if $A \neq \varnothing$ Note that $\chi_A(\mathcal{X}) > 0$ for all $A \in \mathbb{A}$ and $\mathcal{X} \in \mathbb{K}$.

We often need the particular semicharacter

$$\chi_1(\mathcal{X}) := \int_{X^2} \exp(-r_X(x_1, x_2))\, \mu_X^{\otimes 2}(dx) \tag{3.2}$$

defined by taking as $A \in \mathbb{A}$ an array with the single element 1.

As we recalled in the Introduction, a compact metric measure space (X, r_X, μ_X) is uniquely determined by the distribution of the infinite random matrix of distances

$$(r_X(\xi_i, \xi_j))_{(i,j) \in \mathbb{N} \times \mathbb{N}},$$

where $(\xi_k)_{k \in \mathbb{N}}$ is an i.i.d. sample of points in X with common distribution μ_X. The following lemma follows immediately from this observation and the unicity of Laplace transforms.

Lemma 3.2. *Two elements $\mathcal{X}, \mathcal{Y} \in \mathbb{K}$ are equal if and only if $\chi_A(\mathcal{X}) = \chi_A(\mathcal{Y})$ for all $A \in \mathbb{A}$.*

Remark 3.3. Note that if $A' \in \mathbb{R}_+^{\binom{n'}{2}}$ and $A'' \in \mathbb{R}_+^{\binom{n''}{2}}$, then $\chi_{A'}\chi_{A''} = \chi_A$, where $A \in \mathbb{R}_+^{\binom{n'+n''}{2}}$ is given by

$$a_{ij} = \begin{cases} a'_{ij}, & 1 \leqslant i < j \leqslant n' \\ a''_{i-n', j-n'}, & n'+1 \leqslant i < j < n' + n''. \end{cases}$$

It follows that $\{\chi_A : A \in \mathbb{A}\}$ is a semigroup with identity $\chi_\varnothing \equiv 1$.

Remark 3.4. Not all semicharacters of $\mathbb{K}$ are of the form χ_A for some $A \in \mathbb{A}$. For example, if $A \in \mathbb{A}$ and $\beta > 0$, then $\mathcal{X} \mapsto \chi_A(\mathcal{X})^\beta$ is a (continuous) semicharacter. If X has two points, say 0 and 1, that are distance r apart and $\mu_X(\{0\}) = (1-p)$ and $\mu_X(\{1\}) = p$ for some $0 < p < 1$, then taking A to be the array with the single element a we have $\chi_A(\mathcal{X}) = (1-p)^2 + p^2 + 2p(1-p)\exp(-ar)$ and it is not hard to see from considering just $\mathcal{X}$ of this special type that for $\beta \neq 1$ the semicharacter χ_A^β is not of the form $\chi_{A'}$ for some other $A \in \mathbb{A}$.

It follows from part (a) of Lemma 2.4 that $\mathcal{X} \mapsto \exp(-\operatorname{diam}(\mathcal{X}))$ is a (discontinuous) semicharacter on $\mathbb{K}$. Also, if $A \in \mathbb{A}$ and $b > 0$, then

$$\left(\int_{X^n} \exp\left(\sum_{1\leqslant i<j\leqslant n} a_{ij} r_X(x_i,x_j)\right) \mu_X^{\otimes n}(dx)\right)^{-b}$$

is a (discontinuous) semicharacter. These last two examples are connected by the observation that

$$\exp(-\operatorname{diam}(\mathcal{X})) = \lim_{t\to\infty} \left(\int_{X^2} \exp\left(t\, r_X(x_1,x_2)\right) \mu_X^{\otimes 2}(dx)\right)^{-\frac{1}{t}}.$$

Lemma 3.5. *A sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$ converges to $\mathcal{X} \in \mathbb{K}$ if and only if $\lim_{n\to\infty} \chi_A(\mathcal{X}_n) = \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$.*

Proof. For $n \in \mathbb{N}$, let $(\xi_k^{(n)})_{k\in\mathbb{N}}$ be an i.i.d. sequence of X_n-valued random variables with common distribution μ_{X_n}, and let $(\xi_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence of X-valued random variables with common distribution μ_X. It follows from [GPW09, Theorem 5] that $\mathcal{X}_n$ converges to $\mathcal{X}$ if and only if the distribution of $(r_{X_n}(\xi_i^{(n)}, \xi_j^{(n)}))_{1\leqslant i<j\leqslant m}$ converges to that of $(r_X(\xi_i,\xi_j))_{1\leqslant i<j\leqslant m}$ for all $m \in \mathbb{N}$. The result now follows from the equivalence between the weak convergence of probability measures on $\mathbb{R}_+^{\binom{m}{2}}$ and the convergence of their Laplace transforms. □

Corollary 3.6. a) *Suppose that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{K}$ such that $\mathcal{X}_0 \leqslant \mathcal{X}_1 \leqslant \cdots \leqslant \mathcal{Z}$ for some $\mathcal{Z} \in \mathbb{K}$. Then, $\lim_{n\to\infty} \mathcal{X}_n$ exists.*

b) *Suppose that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{K}$ such that $\mathcal{X}_0 \geqslant \mathcal{X}_1 \geqslant \cdots$. Then, $\lim_{n\to\infty} \mathcal{X}_n$ exists.*

Proof. We prove claim (a). The proof of claim (b) is similar, and so we omit it. It follows from Lemma 2.7 that any subsequence of $(\mathcal{X}_n)_{n\in\mathbb{N}}$ has a further subsequence that converges. For any $A \in \mathbb{A}$ the sequence $(\chi_A(\mathcal{X}_n))_{n\in\mathbb{N}}$ is nonincreasing and hence convergent. By Lemma 3.5, all of the convergent subsequences described above converge to the same limit, and so the sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ itself converges to that limit. □

Corollary 3.7. a) *Suppose that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n = \mathcal{Y}$ for some $\mathcal{Y} \in \mathbb{K}$. Suppose further that $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ is a sequence that is obtained by re-ordering the sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$. Then, $\lim_{n\to\infty} \mathcal{X}'_0 \boxplus \cdots \boxplus \mathcal{X}'_n = \mathcal{Y}$ also.*

b) *The limit* $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ *exists if and only if* $\sum_n \operatorname{diam}(\mathcal{X}_n) < \infty$.

Proof. Consider claim (a). For any $A \in \mathbb{A}$, $-\log \chi_A(\mathcal{X}_n) \geqslant 0$ for all $n \in \mathbb{N}$. It follows from Lemma 3.5 that $-\sum_n \log \chi_A(\mathcal{X}_n) = -\log \chi_A(\mathcal{Y})$ It is well-known that all rearrangements of a convergent sequence with nonnegative terms converge to the same limit. Thus, $-\sum_n \log \chi_A(\mathcal{X}'_n) = -\sum_n \log \chi_A(\mathcal{X}_n) = -\log \chi_A(\mathcal{Y})$, implying that $\lim_{n\to\infty} \chi_A(\mathcal{X}'_0 \boxplus \cdots \boxplus \mathcal{X}'_n) = \chi_A(\mathcal{Y})$ and hence, by Lemma 3.5, that $\lim_{n\to\infty} \mathcal{X}'_0 \boxplus \cdots \boxplus \mathcal{X}'_n = \mathcal{Y}$.

Turning to claim (b), suppose that $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n = \mathcal{Y}$. Since $\mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n \leqslant \mathcal{Y}$, $\operatorname{diam}(\mathcal{X}_0) + \cdots + \operatorname{diam}(\mathcal{X}_n) = \operatorname{diam}(\mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n) \leqslant \operatorname{diam}(\mathcal{Y})$, and so $\sum_n \operatorname{diam}(\mathcal{X}_n) < \infty$. Conversely, suppose that $\sum_n \operatorname{diam}(\mathcal{X}_n) < \infty$. For $m < n$ we have from Lemma 2.1 and part (b) of Lemma 2.4 that

$$\begin{aligned} & d_{\mathrm{GPr}}(\mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_m, \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n) \\ & \quad \leqslant d_{\mathrm{GPr}}(\mathcal{E}, \mathcal{X}_{m+1} \boxplus \cdots \boxplus \mathcal{X}_n) \\ & \quad \leqslant \operatorname{diam}(\mathcal{X}_{m+1} \boxplus \cdots \boxplus \mathcal{X}_n) \\ & \quad = \operatorname{diam}(\mathcal{X}_{m+1}) + \cdots + \operatorname{diam}(\mathcal{X}_n). \end{aligned}$$

It follows that the partial sums of $(\mathcal{X}_n)_{n\in\mathbb{N}}$ form a Cauchy sequence and so, by the completeness of $(\mathbb{K}, d_{\mathrm{GPr}})$, $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ exists. $\square$

Remark 3.8. It follows from Corollary 3.7 that if $(\mathcal{X}_s)_{s\in S}$ is a countable collection of elements of $\mathbb{K}$, then the existence of $\lim_{n\to\infty} \mathcal{X}_{s_0} \boxplus \cdots \boxplus \mathcal{X}_{s_n}$ for some listing $(s_n)_{n\in\mathbb{N}}$ implies the existence for any other listing, with the same value for the limit. We will therefore unambiguously denote the limit when it exists by the notation $\boxplus_{s\in S} \mathcal{X}_s$. Moreover, a necessary and sufficient condition for $\boxplus_{s\in S} \mathcal{X}_s$ to exist is that $\sum_{s\in S} \operatorname{diam}(\mathcal{X}_s) < \infty$.

4. Algebraic properties

An element of a semigroup with an identity is a *unit* if it has an inverse and a semigroup with an identity is said to be *reduced* if the only unit is the identity (see [Cli38, Section 1]. The following result is immediate from part (a) of Lemma 2.4.

Lemma 4.1. *The semigroup* $(\mathbb{K}, \boxplus)$ *is reduced.*

In the usual terminology of semigroup theory, part (a) of the following result says that the semigroup $(\mathbb{K}, \boxplus)$ is *cancellative* (see [Gri01, Section II.1.1]).

Proposition 4.2. a) *Suppose that* $\mathcal{X}, \mathcal{Y}, \mathcal{Z}', \mathcal{Z}'' \in \mathbb{K}$ *satisfy* $\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'$ *and* $\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}''$, *then* $\mathcal{Z}' = \mathcal{Z}''$.

b) *Consider sequences* $(\mathcal{X}_n)_{n\in\mathbb{N}}$ *and* $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ *in* $\mathbb{K}$. *Set* $\mathcal{Z}_n := \mathcal{X}_n \boxplus \mathcal{Y}_n$. *Suppose that* $\mathcal{X} := \lim_{n\to\infty} \mathcal{X}_n$ *and* $\mathcal{Z} := \lim_{n\to\infty} \mathcal{Z}_n$ *exist. Then,* $\mathcal{Y} := \lim_{n\to\infty} \mathcal{Y}_n$ *exists and* $\mathcal{Z} = \mathcal{X} \boxplus \mathcal{Y}$.

Proof. a) For each semicharacter χ_A, $A \in \mathbb{A}$, we have $\chi_A(\mathcal{Y})\chi_A(\mathcal{Z}') = \chi_A(\mathcal{X}) = \chi_A(\mathcal{Y})\chi_A(\mathcal{Z}'')$ and so $\chi_A(\mathcal{Z}') = \chi_A(Z'')$, which implies that $\mathcal{Z}' = \mathcal{Z}''$.
b) By Lemma 2.7, the sequence $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ is pre-compact. Any subsequential limit $\mathcal{Y}_\infty$ will satisfy $\mathcal{Z} = \mathcal{X} \boxplus \mathcal{Y}_\infty$. It follows from part (a) that $\mathcal{Y} := \lim_{n\to\infty} \mathcal{Y}_n$ exists and $\mathcal{Z} = \mathcal{X} \boxplus \mathcal{Y}$. □

Remark 4.3. It follows from part (a) of Proposition 4.2 and the discussion in Section 1.10 of [CP61] that the semigroup $(\mathbb{K}, \boxplus)$ can be embedded into a group $\mathbb{G}$ as follows. Equip $\mathbb{K} \times \mathbb{K}$ with the equivalence relation $\equiv$ defined by $(\mathcal{W}, \mathcal{X}) \equiv (\mathcal{Y}, \mathcal{Z})$ if $\mathcal{W} \boxplus \mathcal{Z} = \mathcal{X} \boxplus \mathcal{Y}$. It is not hard to see that $\equiv$ is indeed an equivalence relation, the only property that is not completely obvious is transitivity. However, if $(\mathcal{U}, \mathcal{V}) \equiv (\mathcal{W}, \mathcal{X})$ and $(\mathcal{W}, \mathcal{X}) \equiv (\mathcal{Y}, \mathcal{Z})$, then, by definition, $\mathcal{U} \boxplus \mathcal{X} = \mathcal{V} \boxplus \mathcal{W}$ and $\mathcal{W} \boxplus \mathcal{Z} = \mathcal{X} \boxplus \mathcal{Y}$ so that

$$\begin{aligned}(\mathcal{U} \boxplus \mathcal{Z}) \boxplus (\mathcal{X} \boxplus \mathcal{W}) &= (\mathcal{U} \boxplus \mathcal{X}) \boxplus (\mathcal{W} \boxplus \mathcal{Z}) \\ &= (\mathcal{V} \boxplus \mathcal{W}) \boxplus (\mathcal{X} \boxplus \mathcal{Y}) = (\mathcal{V} \boxplus \mathcal{Y}) \boxplus (\mathcal{X} \boxplus \mathcal{W}),\end{aligned}$$

from which it follows that $\mathcal{U} \boxplus \mathcal{Z} = \mathcal{V} \boxplus \mathcal{Y}$ and hence $(\mathcal{U}, \mathcal{V}) \equiv (\mathcal{Y}, \mathcal{Z})$. The elements of the group $\mathbb{G}$ are the equivalence classes for this relation. We write $\boxplus$ for the binary operation on $\mathbb{G}$ and define it to be the operation that takes the equivalence classes of $(\mathcal{W}, \mathcal{X})$ and $(\mathcal{Y}, \mathcal{Z})$ to the equivalence class of $(\mathcal{W} \boxplus \mathcal{Y}, \mathcal{X} \boxplus \mathcal{Z})$. It is clear that this operation is well-defined, associative and commutative. The identity element is the equivalence class of $(\mathcal{E}, \mathcal{E})$ and the inverse of the equivalence class of $(\mathcal{Y}, \mathcal{Z})$ is the equivalence class of $(\mathcal{Z}, \mathcal{Y})$.

In the following result we use the notation $\mathcal{V}^{\boxplus n}$ for $\mathcal{V} \in \mathbb{K}$ and $n \in \mathbb{N}$ to denote $\mathcal{V} \boxplus \cdots \boxplus \mathcal{V}$, where there are n terms and we adopt the convention that this quantity is $\mathcal{E}$ for $n = 0$.

Corollary 4.4. a) *For all* $n \in \mathbb{N}$, *the set* $\{(\mathcal{X}, \mathcal{Y}) \in \mathbb{K}^2 : \mathcal{Y}^{\boxplus n} \leqslant \mathcal{X}\}$ *is closed.*

b) *The function* $M : \mathbb{K}^2 \to \mathbb{N}$ *defined by* $M(\mathcal{X}, \mathcal{Y}) = \max\{n \in \mathbb{N} : \mathcal{Y}^{\boxplus n} \leqslant \mathcal{X}\}$ *is upper semicontinuous and hence Borel.*

Proof. Part (a) is immediate from Proposition 4.2. For part (b), $\{(\mathcal{X}, \mathcal{Y}) \in \mathbb{K}^2 : M(\mathcal{X}, \mathcal{Y}) \geqslant n\} = \{(\mathcal{X}, \mathcal{Y}) \in \mathbb{K}^2 : \mathcal{Y}^{\boxplus n} \leqslant \mathcal{X}\}$ is a closed set for all $n \in \mathbb{N}$ by part (a), and this is equivalent to the upper semicontinuity of M. $\square$

5. Arithmetic properties

An element $\mathcal{X} \in \mathbb{K}$ is *irreducible* if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{Y} \leqslant \mathcal{X}$ for $\mathcal{Y} \in \mathbb{K}$ implies that $\mathcal{Y}$ is either $\mathcal{E}$ or $\mathcal{X}$ (see [Cli38, Section 1]).

We write $\mathbb{I}$ for the set of irreducible elements of $\mathbb{K}$. It is not clear *a priori* that $\mathbb{I}$ is nonempty. For example, the semigroup $\mathbb{R}_+$ with the usual addition operation has no irreducible elements in the sense of the general definition in [Cli38]. The following two results show that $\mathbb{I}$ is certainly nonempty.

Proposition 5.1. *The set $\mathbb{I}$ is a dense, G_δ subset of $\mathbb{K}$.*

Proof. We first show that $\mathbb{I}$ is dense in $\mathbb{K}$ As in the proof of [GPW09, Proposition 5.6], the subset of $\mathbb{F} \subseteq \mathbb{K}$ consisting of compact metric measure spaces with finitely many points is dense in $\mathbb{K}$. If we are given a finite metric measure space (W, r_W, μ_W), then convergence of a sequence of probability measures in the Prohorov metric on (W, r_W) is just pointwise convergence of the probabilities assigned to each point of W. The set of probability measures that assign positive probability to all points of W is thus just the relative interior of the $(\#W - 1)$-dimensional simplex thought of as a subset of $\mathbb{R}^{\#W}$ equipped with the usual Euclidean topology. Suppose that (W, r_W) is isometric to $(U \times V, r_U \otimes r_V)$ for some nontrivial finite compact metric spaces (U, r_U) and (V, r_V) – if this is not the case, then (W, r_W, μ_w) is already irreducible. The probability measures on $U \times V$ that are of the form $\mu_U \otimes \mu_V$ form a $(\#U - 1) + (\#V - 1)$-dimensional surface in the $(\#U \times \#V - 1)$-dimensional simplex of probability measures on $U \times V$ and, in particular, the former set is nowhere dense. Thus, even if (W, r_W) is isometric to $(U \times V, r_U \otimes r_V)$, any probability measure on W that is the isometric image of a probability measure on $U \times V$ of the form $\mu_U \otimes \mu_V$ is arbitrarily close to probability measures on W that are not isometric images of probability measures of this form, and it follows that $\mathbb{I}$ is dense in $\mathbb{K}$.

We know show that the set $\mathbb{I}$ is a G_δ. This is equivalent to showing that $\mathbb{K} \backslash \mathbb{I}$ is an F_σ.

Let $\chi := \chi_1$ be the semicharacter defined by (3.2). Recall that $\chi_1(\mathcal{X}) = 1$ if and only if $\mathcal{X} = \mathcal{E}$. For $0 < \varepsilon < \frac{1}{2}$ set

$$\mathbb{L}_\varepsilon := \{\mathcal{X} \in \mathbb{K} : \exists \mathcal{Y} \leqslant \mathcal{X},\ \chi(\mathcal{X})^{1-\varepsilon} \leqslant \chi(\mathcal{Y}) \leqslant \chi(\mathcal{X})^\varepsilon\}.$$

Note that $\mathbb{L}_{\varepsilon'} \supseteq \mathbb{L}_{\varepsilon''}$ for $\varepsilon' \leqslant \varepsilon''$ and $\bigcup_{0<\varepsilon<\frac{1}{2}} \mathbb{L}_\varepsilon = \mathbb{K} \backslash \mathbb{I}$, so it suffices to show that the $\mathbb{L}_\varepsilon$ are closed. Suppose that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{L}_\varepsilon$ that converges to $\mathcal{X} \in \mathbb{K}$. For each $n \in \mathbb{N}$ there exist $\mathcal{Y}_n$ and $\mathcal{Z}_n$ in $\mathbb{K}$ such that $\mathcal{X}_n = \mathcal{Y}_n \boxplus \mathcal{Z}_n$ and $\chi(\mathcal{X}_n)^{1-\varepsilon} \leqslant \chi(\mathcal{Y}_n) \leqslant \chi(\mathcal{X}_n)^\varepsilon$. By Lemma 2.7 and part (b) of Proposition 4.2, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mathcal{Y}_{n_k} = \mathcal{Y}$ and $\lim_{k\to\infty} \mathcal{Z}_{n_k} = \mathcal{Z}$ for $\mathcal{Y}, \mathcal{Z} \in \mathbb{K}$ such that $\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}$. Thus, $\mathcal{Y} \leqslant \mathcal{X}$ and $\chi(\mathcal{X})^{1-\varepsilon} \leqslant \chi(\mathcal{Y}) \leqslant \chi(\mathcal{X})^\varepsilon$, so that $\mathcal{X} \in \mathbb{L}_\varepsilon$, as required. $\square$

In particular, the space $\mathbb{I}$ with the relative topology inherited from $\mathbb{K}$ is a Polish space. This follows follows from Alexandrov's theorem saying that a subspace of a Polish space is Polish in the relative topology if and only if it is a G_δ-set, see [Kec95, Theorem 3.11].

Proposition 5.2. *Given any $\mathcal{X} \in \mathbb{K}\backslash\{\mathcal{E}\}$, there exists $\mathcal{Y} \in \mathbb{I}$ with $\mathcal{Y} \leqslant \mathcal{X}$.*

Proof. Define $\Gamma : \mathbb{K} \to \mathbb{R}_+$ by

$$\Gamma(\mathcal{Z}) := \int_{Z\times Z} r_Z(z', z'')\, \mu_Z^{\otimes 2}(dz', dz'').$$

Note that the function Γ is continuous on the compact set $\{\mathcal{Z} \in \mathbb{K} : \mathcal{Z} \leqslant \mathcal{X}\}$, $\Gamma(\mathcal{Z}') + \Gamma(\mathcal{Z}'') = \Gamma(\mathcal{Z}' \boxplus \mathcal{Z}'')$, and $\Gamma(\mathcal{Z}')^2 + \Gamma(\mathcal{Z}'')^2 \leqslant \Gamma(\mathcal{Z}' \boxplus \mathcal{Z}'')^2$, with strict inequality unless $\mathcal{Z}' = \mathcal{E}$ or $\mathcal{Z}'' = \mathcal{E}$.

Let L_k be the set of binary strings of length k, where for $k = 0$ we denote the empty string by $\varnothing$. Set $\mathcal{X}_\varnothing = \mathcal{X}$. Suppose that $\mathcal{X}_w \in \mathbb{K}$ have been defined for $w \in L_k$ where $0 \leqslant k \leqslant n$. Choose $\mathcal{X}_w$ for $w \in L_{n+1}$ so that $\mathcal{X}_{v0} \boxplus \mathcal{X}_{v1} = \mathcal{X}_v$ for all $v \in L_n$ and $\Gamma(X_{v0})^2 + \Gamma(X_{v1})^2$ is minimized subject to this requirement. This is possible by Lemma 2.7 and the continuity of Γ.

It cannot be the case that $\lim_{n\to\infty} \max_{w\in L_n} \Gamma(\mathcal{X}_w) = 0$, because it would then follow from [Fel71, Section XVII.7] that the image of $\mu_X^{\otimes 2}$ under the map $(x', x'') \to r_X(x', x'')$ would be a nontrivial infinitely divisible probability measure that is supported on $[0, \operatorname{diam}(\mathcal{X})]$, contradicting the fact that all nontrivial infinitely divisible probability measures have unbounded support. (The last fact is immediate from the Itô representation of a nonnegative infinitely divisible random variable as $c + \int x\, \Pi(dx)$, where c is a constant and Π is a Poisson random measure on $\mathbb{R}_{++}$ with intensity measure ν that satisfies $\int (x \wedge 1)\, \nu(dx) < \infty$. See, also, Remark 5.3.)

It follows that there is a sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in L_n$ such that $\Gamma(\mathcal{X}) = \Gamma(\mathcal{X}_{w_0}) \geqslant \Gamma(\mathcal{X}_{w_1}) \geqslant \Gamma(\mathcal{X}_{w_2}) \geqslant \cdots \geqslant \gamma$ for some constant $\gamma > 0$. Infinitely many of the w_n must be either of the form $0v_n$ or $1v_n$ for some $v_n \in L_{n-1}$, so we can pick $\sigma_1 \in \{0, 1\}$ such that infinitely many of the w_n are of the form $\sigma_1 v_n$ for some $v_n \in L_{n-1}$. Similarly, infinitely many of the w_n must be either of the form $\sigma_1 0 u_n$ or $\sigma_1 1 u_n$

for $u_n \in L_{n-2}$, so we can pick $\sigma_2 \in \{0,1\}$ such that infinitely many of the w_n are of the form $\sigma_1\sigma_2 u_n$ for some $u_n \in L_{n-2}$. Continuing in this we, we can find $\sigma_1, \sigma_2, \ldots \in \{0,1\}$ such that $\mathcal{X} = \mathcal{X}_\varnothing \geqslant \mathcal{X}_{\sigma_1} \geqslant \mathcal{X}_{\sigma_1\sigma_2} \geqslant \cdots \geqslant \mathcal{X}_{\sigma_1\cdots\sigma_n} \geqslant \cdots$ and $\Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_n}) \geqslant \gamma$.

By Corollary 3.6 there exists $\mathcal{Y} \in \mathbb{K}$ with $\Gamma(\mathcal{Y}) \geqslant \gamma > 0$ such that $\lim_{n\to\infty} \mathcal{X}_{\sigma_1\cdots\sigma_n} = \mathcal{Y}$.

We claim that $\mathcal{Y}$ must be irreducible. If this is not so, then $\mathcal{Y} = \mathcal{Y}' \boxplus \mathcal{Y}''$, where neither $\mathcal{Y}'$ nor $\mathcal{Y}''$ is $\mathcal{E}$. By Proposition 4.2, we can write $\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\sigma_n} = \mathcal{Y}' \boxplus \mathcal{Y}'' \boxplus \mathcal{Z}_n$ for some unique $\mathcal{Z}_n \in \mathbb{K}$ such that $\lim_{n\to\infty} \mathcal{Z}_n = \mathcal{E}$. Set $\bar{0} := 1$ and $\bar{1} := 0$. Note that

$$\begin{aligned}&\Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\sigma_n})^2 + \Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n})^2\\ &\quad = (\Gamma(\mathcal{Y}') + \Gamma(\mathcal{Y}'') + \Gamma(\mathcal{Z}_n))^2 + (\Gamma(\mathcal{Z}_{n-1}) - \Gamma(\mathcal{Z}_n))^2\end{aligned}$$

If we define $\tilde{\mathcal{X}}_w$, $w \in L_n$, by

$$\tilde{\mathcal{X}}_w := \mathcal{X}_w, \quad w \notin \{\sigma_1\cdots\sigma_{n-1}\sigma_n, \sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n\},$$

$$\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\sigma_n} := \mathcal{Y}' \boxplus \mathcal{Z}_n$$

and

$$\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n} := \mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n} \boxplus \mathcal{Y}'',$$

then $\mathcal{X}_v = \tilde{\mathcal{X}}_{v0} \boxplus \tilde{\mathcal{X}}_{v1}$ for all $v \in L_{n-1}$ and

$$\begin{aligned}&\Gamma(\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\sigma_n})^2 + \Gamma(\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n})^2\\ &\quad = (\Gamma(\mathcal{Y}') + \Gamma(\mathcal{Z}_n))^2 + (\Gamma(\mathcal{Z}_{n-1}) - \Gamma(\mathcal{Z}_n) + \Gamma(\mathcal{Y}''))^2.\end{aligned}$$

Thus, $\Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\sigma_n})^2 + \Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n})^2$ becomes arbitrarily close to $(\Gamma(\mathcal{Y}') + \Gamma(\mathcal{Y}''))^2$ for sufficiently large n, whereas $\Gamma(\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\sigma_n})^2 + \Gamma(\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n})^2$ can be made arbitrarily close to $\Gamma(\mathcal{Y}')^2 + \Gamma(\mathcal{Y}'')^2$ by taking n sufficiently large. Since $\Gamma(\mathcal{Y}')^2 + \Gamma(\mathcal{Y}'')^2 < (\Gamma(\mathcal{Y}') + \Gamma(\mathcal{Y}''))^2$, we would have

$$\begin{aligned}&\Gamma(\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\sigma_n})^2 + \Gamma(\tilde{\mathcal{X}}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n})^2\\ &\quad < \Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\sigma_n})^2 + \Gamma(\mathcal{X}_{\sigma_1\cdots\sigma_{n-1}\bar{\sigma}_n})^2\end{aligned}$$

for n sufficiently large, which violates the definition of $(\mathcal{X}_w)_{w\in L_n}$. □

Remark 5.3. In the proof of Proposition 5.2 we used the classical limit theory for triangular arrays of random variables to establish the following fact. If ζ is a bounded, nonnegative random variable that can be written as $\sum_j \xi_{nj}$ for all $n \in \mathbb{N}$, where the random variables $\xi_{n1}, \xi_{n2}, \ldots$ are nonnegative, independent and satisfy $\lim_{n\to\infty} \sup_j \mathbb{E}[\xi_{nj}] = 0$, then

ζ is almost surely constant. This result can be proved directly as follows. Note from a Taylor expansion that $\mathbb{E}[\exp(-\xi_{nj})] \leqslant 1-c_n\mathbb{E}[\xi_{nj}] \leqslant \exp(-c_n\mathbb{E}[\xi_{nj}])$ for constants $(c_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} c_n = 1$. Thus,

$$\begin{aligned}\mathbb{E}[\exp(-\zeta)] &= \prod_j \mathbb{E}[\exp(-\xi_{nj})] \\ &\leqslant \prod_j \exp(-c_n\mathbb{E}[\xi_{nj}]) \\ &= \exp(-c_n\mathbb{E}[\zeta]) \\ &\to \exp(-\mathbb{E}[\zeta]), \quad n\to\infty.\end{aligned}$$

Jensen's inequality gives the opposite inequality $\exp(-\mathbb{E}[\zeta]) \leqslant \mathbb{E}[\exp(-\zeta)]$, with equality if and only if ζ is almost surely constant, and so ζ must indeed be almost surely constant.

Remark 5.4. It is not difficult to construct concrete examples of irreducible elements of $\mathbb{K}$.

We first recall that a metric space (W, r_W) is *totally geodesic* if for any pair of points $w', w'' \in W$ there is a unique map $\phi : [0, r_W(w', w'')] \to W$ such that $\phi(0) = w'$, $\phi(r_W(w', w'')) = w''$ and $r_W(\phi(s), \phi(t)) = |s-t|$ for $s, t \in [0, r_W(w', w'')]$; that is, any two points of W are joined by a unique geodesic segment.

Any nontrivial compact subset X of a totally geodesic metric space W is irreducible no matter what measure it is equipped with because such a space (X, r_W) cannot be isometric to a space of the form $(Y \times Z, r_Y \bigoplus r_Z)$ for nontrivial Y and Z. To see this, suppose that the claim is false. There will then be four distinct points a, b, c, d in X that are isometric images of points of the form (y', z'), (y'', z'), (y', z''), (y'', z'') in $Y \times Z$. Suppose that (X, r_W) is a compact subset of the totally geodesic space (W, r_W). We have

$$\begin{aligned}r_W(a,b) &= r_W(c,d), \\ r_W(a,c) &= r_W(b,d), \\ r_W(a,d) &= r_W(a,b) + r_W(b,d), \\ r_W(a,d) &= r_W(a,c) + r_W(c,d), \\ r_W(b,c) &= r_W(a,b) + r_W(c,a),\end{aligned}$$

and

$$r_W(b,c) = r_W(b,d) + r_W(c,d).$$

It follows from the third and fourth equalities that b and c are on the geodesic segment between a and d. We may therefore suppose that (W, r_W) is a closed subinterval of $\mathbb{R}$ and, without loss of generality, that $a < b < c < d$. The fifth and sixth equalities are then impossible.

There are many totally geodesic metric spaces. A Banach space $(X, \| \|)$ is totally geodesic if and only if it is *strictly convex*; that is, $x \neq y$ and $\|x'\| = \|x''\| = 1$ imply that $\|ax' + (1-a)x''\| < 1$ for all $0 < a < 1$ [Bea85, Section 3.I.1]. Strict convexity of $(X, \| \|)$ is implied by *uniform convexity*; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x'\| = \|x''\| = 1$ and $\|x' - x''\| \geqslant \varepsilon$ imply $\|\frac{x'+x''}{2}\| \leqslant 1 - \delta$. Any Hilbert space is uniformly convex and the Banach spaces $L^p(S, \mathcal{S}, \lambda)$, $1 < p < \infty$, where λ is a σ-finite measure, are uniformly convex [Bea85, Section 3.II.1]. Also, any real tree is, by definition, totally geodesic and any compact ultrametric space is isometric to a compact subset of a real tree.

The prime numbers are the analogue of irreducible elements for the semigroup of positive integers equipped with the usual multiplication. The key to proving the Fundamental Theorem of Arithmetic (that every positive integer other than 1 has a factorization into primes that is unique up to the order of the factors) is a lemma due to Euclid which says that if a prime number number divides the product of two positive integers, then it must divide one of the factors. For general commutative semigroups, the term "prime" is usually reserved for elements that exhibit the generalization of this property (see, for example, [Cli38]). Accordingly, we say that an element $\mathcal{X} \in \mathbb{K} \backslash \{\mathcal{E}\}$ is *prime* if $\mathcal{X} \leqslant \mathcal{Y} \boxplus \mathcal{Z}$ for $\mathcal{Y}, \mathcal{Z} \in \mathbb{K}$ implies that $\mathcal{X} \leqslant \mathcal{Y}$ or $\mathcal{X} \leqslant \mathcal{Z}$. Prime elements are clearly irreducible, but the converse is not *a priori* true and there are commutative, cancellative semigroups where the analogue of the converse is false.

Before showing that the notions of irreducibility and primality coincide in our setting, we need the following elementary lemma which we prove for the sake of completeness.

Lemma 5.5. *Let $\xi_{00}, \xi_{01}, \xi_{10}, \xi_{11}$ be random elements of the respective compact metric spaces $X_{00}, X_{01}, X_{10}, X_{11}$. Suppose that the pairs (ξ_{00}, ξ_{01}) and (ξ_{10}, ξ_{11}) are independent and that the pairs (ξ_{00}, ξ_{10}) and (ξ_{01}, ξ_{11}) are independent. Then, $\xi_{00}, \xi_{01}, \xi_{10}, \xi_{11}$ are independent.*

Proof. Suppose that $f_{ij} : X_{ij} \to \mathbb{R}$, $i, j \in \{0, 1\}$, are bounded Borel functions. Using first the independence of (ξ_{00}, ξ_{01}) and (ξ_{10}, ξ_{11}), and then the independence of (ξ_{00}, ξ_{10}) and (ξ_{01}, ξ_{11}), we have

$$\begin{aligned}
&\mathbb{E}[f_{00}(\xi_{00}) f_{01}(\xi_{01}) f_{10}(\xi_{10}) f_{11}(\xi_{11})] \\
&\quad = \mathbb{E}[f_{00}(\xi_{00}) f_{01}(\xi_{01})] \mathbb{E}[f_{10}(\xi_{10}) f_{11}(\xi_{11})] \\
&\quad = \mathbb{E}[f_{00}(\xi_{00})] \mathbb{E}[f_{01}(\xi_{01})] \mathbb{E}[f_{10}(\xi_{10})] \mathbb{E}[f_{11}(\xi_{11})],
\end{aligned}$$

as required. $\square$

Proposition 5.6. *All irreducible elements of* $\mathbb{K}$ *are prime. Moreover, if* $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ *is a sequence of elements of* $\mathbb{K}$ *such that* $\lim_{n\to\infty} \mathcal{Y}_0 \boxplus \cdots \boxplus \mathcal{Y}_n = \mathcal{Y}$ *exists and* $\mathcal{X} \in \mathbb{I}$ *is such that* $\mathcal{X} \leqslant \mathcal{Y}$*, then* $\mathcal{X} \leqslant \mathcal{Y}_n$ *for some* $n \in \mathbb{N}$.

Proof. Consider the first claim. Suppose that $\mathcal{X} \in \mathbb{K}$ is irreducible and $\mathcal{X} \leqslant \mathcal{Y} \boxplus \mathcal{Z}$ for some $\mathcal{Y}, \mathcal{Z} \in \mathbb{K}$.

From Proposition 4.2 we have $\mathcal{Y} \boxplus \mathcal{Z} = \mathcal{W} \boxplus \mathcal{X}$ for some unique $\mathcal{W} \in \mathbb{K}$. From the remarks at the end of [Tar92], we may suppose that there are compact metric spaces $(Y', r_{Y'})$, $(X', r_{X'})$, $(X'', r_{X''})$ and $(Z'', r_{Z''})$ such that $(Y, r_Y) = (Y' \times X', r_{Y'} \oplus r_{X'})$, $(Z, r_Z) = (X'' \times Z'', r_{X''} \oplus r_{Z''})$, $(X, r_X) = (X' \times X'', r_{X'} \oplus r_{X''})$ and $(W, r_W) = (Y' \times Z'', r_{Y'} \oplus r_{Z''})$, so that $(Y \times Z, r_Y \oplus r_Z) = (W \times X, r_W \oplus r_X) = (Y' \times X' \times X'' \times Z'', r_{Y'} \oplus r_{X'} \oplus r_{X''} \oplus r_{Z''})$ (see also [Wal87] for an analogous result concerning the existence of a common refinement of two Cartesian factorizations of a (possibly infinite) graph and [AFDF00] for the case of finite metric spaces). It follows from Lemma 5.5 that there are probability measures $\mu_{Y'}$, $\mu_{X'}$, $\mu_{X''}$ and $\mu_{Z''}$ such that $\mu_Y = \mu_{Y'} \otimes \mu_{X'}$, $\mu_Z = \mu_{X''} \otimes \mu_{Z''}$, $\mu_X = \mu_{X'} \otimes \mu_{X''}$, $\mu_W = \mu_{Y'} \otimes \mu_{Z''}$, and $\mu_Y \otimes \mu_Z = \mu_W \otimes \mu_X = \mu_{Y'} \otimes \mu_{X'} \otimes \mu_{X''} \otimes \mu_{Z''}$. Thus, $\mathcal{Y} = \mathcal{Y}' \boxplus \mathcal{X}''$, $\mathcal{Z} = \mathcal{X}' \boxplus \mathcal{Z}''$, $\mathcal{X} = \mathcal{X}' \boxplus \mathcal{X}''$, $\mathcal{W} = \mathcal{Y}' \boxplus \mathcal{Z}''$, and $\mathcal{Y} \boxplus \mathcal{Z} = \mathcal{W} \boxplus \mathcal{X} = \mathcal{Y}' \boxplus \mathcal{X}' \boxplus \mathcal{X}'' \boxplus \mathcal{Z}''$. This contradicts the irreducibility of $\mathcal{X}$ unless $\mathcal{X}' = \mathcal{E}$ or $\mathcal{X}'' = \mathcal{E}$, in which case $\mathcal{X} \leqslant \mathcal{Z}$ or $\mathcal{X} \leqslant \mathcal{Y}$, thus establishing the first claim of the lemma.

Turning to the second claim, let $(\mathcal{Y}_n)_{n\in\mathbb{N}}$, $\mathcal{Y} \in \mathbb{K}$ and $\mathcal{X} \in \mathbb{I}$ satisfy the hypotheses of the claim. By Proposition 4.2, for each $n \in \mathbb{N}$ we have $\mathcal{Y} = \mathcal{Y}_0 \boxplus \cdots \boxplus \mathcal{Y}_n \boxplus \mathcal{Z}_n$ for some unique $\mathcal{Z}_n \in \mathbb{K}$. If there is no $n \in \mathbb{N}$ such that $\mathcal{X} \leqslant \mathcal{Y}_n$, then, by the first part of the lemma, $\mathcal{X} \leqslant \mathcal{Z}_n$ for all $n \in \mathbb{N}$. By Proposition 4.2, this means that $\mathcal{Z}_n = \mathcal{X} \boxplus \mathcal{W}_n$ for some unique $\mathcal{W}_n \in \mathbb{K}$ and hence $\chi_A(\mathcal{Z}_n) \leqslant \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$. However, $\lim_{n\to\infty} \chi_A(\mathcal{Y}_0 \boxplus \cdots \boxplus \mathcal{Y}_n) = \chi_A(\mathcal{Y})$ for all $A \in \mathbb{A}$ and so $\lim_{n\to\infty} \chi_A(\mathcal{Z}_n) = 1$ for all $A \in \mathbb{A}$, implying that $\chi_A(\mathcal{X}) = 1$ for all $A \in \mathbb{A}$. This, however, is impossible, since it would imply that $\mathcal{X} = \mathcal{E} \notin \mathbb{I}$. □

The following is standard, but we include it for the sake of completeness.

Corollary 5.7. *Suppose for* $\mathcal{X} \in \mathcal{K}$ *and distinct* $\mathcal{Y}_1, \ldots, \mathcal{Y}_n \in \mathbb{I}$ *that* $\mathcal{Y}_k \leqslant \mathcal{X}$ *for* $k = 1, \ldots, n$*. Then,* $\mathcal{Y}_1 \boxplus \cdots \boxplus \mathcal{Y}_n \leqslant \mathcal{X}$*.*

Proof. The proof is by induction. The statement is certainly true for $n = 1$. Suppose it is true for $n = r$ and consider the case $n = r + 1$. We have $\mathcal{X} = \mathcal{Y}_1 \boxplus \cdots \boxplus \mathcal{Y}_r \boxplus \mathcal{W}_r$ for some $\mathcal{W}_r \in \mathbb{K}$ by the inductive assumption. Because $\mathcal{Y}_{r+1} \leqslant \mathcal{X} = \mathcal{Y}_1 \boxplus \cdots \boxplus \mathcal{Y}_r \boxplus \mathcal{W}_r$, it follows from Proposition 5.6 that either $\mathcal{Y}_{r+1} \leqslant \mathcal{Y}_k$ for some k with $1 \leqslant k \leqslant r$ or $\mathcal{Y}_{r+1} \leqslant \mathcal{W}_r$. The former alternative is impossible because $\mathcal{Y}_1, \ldots, \mathcal{Y}_r, \mathcal{Y}_{r+1} \in \mathbb{I}$ are distinct. Thus, $\mathcal{Y}_{r+1} \leqslant \mathcal{W}_r$ and we have $\mathcal{W}_r = \mathcal{Y}_{r+1} \boxplus \mathcal{W}_{r+1}$ for some $\mathcal{W}_{r+1} \in \mathbb{K}$. This implies that $\mathcal{X} = \mathcal{Y}_1 \boxplus \cdots \boxplus \mathcal{Y}_r \boxplus \mathcal{Y}_{r+1} \boxplus \mathcal{W}_{r+1}$ and hence $\mathcal{Y}_1 \boxplus \cdots \boxplus \mathcal{Y}_r \boxplus \mathcal{Y}_{r+1} \leqslant \mathcal{X}$, completing the inductive step. □

Theorem 5.8. *Given any $\mathcal{X} \in \mathbb{K}\backslash\{\mathcal{E}\}$, there is either a finite sequence $(\mathcal{X}_n)_{n=0}^N$ or an infinite sequence $(\mathcal{X}_n)_{n=0}^\infty$ of irreducible elements of $\mathbb{K}$ such that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_N$ in the first case and $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ in the second. The sequence is unique up to the order of its terms. Each irreducible element appears a finite number of times, so the representation is specified by the irreducible elements that appear and their finite multiplicities.*

Proof. We first establish the existence claim. Let $\chi := \chi_1$ be the semicharacter defined by (3.2). Put $\mathbb{J}_k := \{\mathcal{Y} \in \mathbb{I} : \mathcal{Y} \leqslant \mathcal{X},\ 1 - 2^{-k} < \chi(\mathcal{Y}) \leqslant 1 - 2^{-(k+1)}\}$ for $k \in \mathbb{N}$. It follows from Proposition 5.6 that if $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$ are distinct elements of $\mathbb{J}_k$, then $\mathcal{Y}_1 \boxplus \cdots \boxplus \mathcal{Y}_m \leqslant \mathcal{X}$, and hence $0 < \chi(\mathcal{X}) \leqslant \chi(\mathcal{Y}_1) \cdots \chi(\mathcal{Y}_m) \leqslant (1 - 2^{-(k+1)})^m$, so that $\mathbb{J}_k$ is finite. Each of the sets $\mathbb{J}_k$ can be ordered. With a slight abuse of notation, we will use the same symbol $\prec$ for this order for all k.

Define $\mathcal{X}_0, \mathcal{X}_1, \ldots$ as follows. Let $K_0 := \min\{k \in \mathbb{N} : \mathbb{J}_k \neq \emptyset\}$ and set $\mathcal{X}_0$ to be minimal element with respect to the order $\prec$ of the set $\{\mathcal{Y} \in \mathbb{J}_{K_0} : \mathcal{Y} \leqslant \mathcal{X}\}$. It follows from Proposition 5.2 that K_0 and $\mathcal{X}_0$ are well-defined. By Proposition 4.2 there exists $\mathcal{Z}_0 \in \mathbb{K}$ such that $\mathcal{X} = \mathcal{X}_0 \boxplus \mathcal{Z}_0$. If $\mathcal{Z}_0 = \mathcal{E}$, then set $N = 0$ and terminate the procedure. Suppose that $\mathcal{X}_0, \ldots, \mathcal{X}_n$ and $\mathcal{Z}_0, \ldots, \mathcal{Z}_n$ have been defined such that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_i \boxplus \mathcal{Z}_i$ for $0 \leqslant i \leqslant n$, where $\mathcal{X}_i \in \mathbb{J}_{K_i}$ and $\mathcal{Z}_i \neq \mathcal{E}$ for $0 \leqslant i \leqslant n$, and the procedure has yet to terminate. Let $K_{n+1} := \inf\{k \in \mathbb{N} : \exists \mathcal{Y} \in \mathbb{J}_k \text{ such that } \mathcal{Y} \leqslant \mathcal{Z}_n\}$ and set $\mathcal{X}_{n+1}$ to be the minimal element with respect to the order $\prec$ of the set $\{\mathcal{Y} \in \mathbb{J}_{K_{n+1}} : \mathcal{Y} \leqslant \mathcal{Z}_n\}$. It follows from Proposition 5.2 that K_{n+1} and $\mathcal{X}_{n+1}$ are well-defined. By Proposition 4.2 there exists $\mathcal{Z}_{n+1} \in \mathbb{K}$ such that $\mathcal{Z}_n = \mathcal{X}_{n+1} \boxplus \mathcal{Z}_{n+1}$, so that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_{n+1} \boxplus \mathcal{Z}_{n+1}$. If $\mathcal{Z}_{n+1} = \mathcal{E}$, then set $N = n + 1$ and terminate the procedure.

If the procedure terminates, then it is clear that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_N$. Suppose that the procedure does not terminate. By part (a) of Corollary 3.7, the sequence $(\mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n)_{n\in\mathbb{N}}$ converges to a limit, say $\mathcal{Y} \leqslant \mathcal{X}$. Observe that $K_0 \leqslant K_1 \leqslant \ldots$. Moreover, $\#\{n : K_n = k\}$ is finite for all $k \in \mathbb{N}$, because otherwise $\lim_{n\to\infty} \chi(\mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n) = 0 \neq \chi(\mathcal{Y})$. Therefore, $K_n \to \infty$ as $n \to \infty$. By Proposition 4.2, $\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}$

for some $\mathcal{Z} \in \mathbb{K}$. If $\mathcal{Y} \neq \mathcal{X}$, so that $\mathcal{Z} \neq \mathcal{E}$, then it follows from Proposition 5.2 that there is a $\mathcal{W} \in \mathbb{I}$ such that $\mathcal{W} \leqslant \mathcal{Z}$. However, this would mean that $\mathcal{W} \leqslant \mathcal{Z}_n$ for all n, but $\mathcal{W} \in \mathbb{J}_k$ for some $k \in \mathbb{N}$, and so this would contradict the conclusion that $K_n \to \infty$ as $n \to \infty$. Therefore, $\mathcal{Y} = \mathcal{X}$, as required.

We now turn to the uniqueness claim. This may fail because $\mathcal{X}$ has two different representations as a finite sum of irreducible elements, one representation as a finite sum and another as a limit of finite sums, or two different representations as a limit of finite sums. We deal with the last case. The other two are similar and are left to the reader. Suppose then that two sequences $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ and $(\mathcal{X}''_n)_{n\in\mathbb{N}}$ represent $\mathcal{X}$. An argument similar to one above shows that any particular irreducible element appears a finite number of times in each sequence. Suppose that $\mathcal{Y} \in \mathbb{I}$ appears M' times in $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ and M'' times in $(\mathcal{X}''_n)_{n\in\mathbb{N}}$ with $M' \neq M''$. Assume without loss of generality that $M' > M''$. We have $\mathcal{Y} \boxplus \cdots \boxplus \mathcal{Y} \boxplus \mathcal{Z}'_n = \mathcal{X} = \mathcal{Y} \boxplus \cdots \boxplus \mathcal{Y} \boxplus \mathcal{Z}''_n$, where the first sum has M' terms, the second sum has M'' terms and $\mathcal{Z}'_n, \mathcal{Z}''_n \in \mathbb{K}$ are such that $\mathcal{Y} \nleqslant \mathcal{Z}'_n$ and $\mathcal{Y} \nleqslant \mathcal{Z}''_n$. Using Proposition 4.2, $\mathcal{Y} \boxplus \cdots \boxplus \mathcal{Y} \boxplus \mathcal{Z}'_n = \mathcal{Z}''_n$, where the sum has $M' - M'' > 0$ terms. This, however, would violate the second part of Proposition 5.6. □

Remark 5.9. It follows easily from Theorem 5.8 that, for the partial order $\leqslant$, every pair of elements of $\mathbb{K}$ has a *join* (that is, a least upper bound) and a *meet* (that is, a greatest lower bound), and so $\mathbb{K}$ with these operations is a *lattice*. It is not hard to check that this lattice is distributive (that is, the meet operation distributes over the join operation and vice versa). Furthermore, the Gromov-Prohorov distance between $\mathcal{X}$ and $\mathcal{Y}$ equals the maximum of the distances between the meet of $\mathcal{X}$ and $\mathcal{Y}$ and either $\mathcal{X}$ or $\mathcal{Y}$.

Remark 5.10. Given $f : \mathbb{I} \to [0,1]$, the map $\chi : \mathbb{K} \to [0,1]$ that sends $\mathcal{X}$ to $\prod_n f(\mathcal{X}_n)$, where $\mathcal{X}_0, \mathcal{X}_1, \ldots$ are as in Theorem 5.8, is a semicharacter.

The following result will be a key ingredient in the characterization of the infinitely divisible random elements of $\mathbb{K}$ in Theorem 9.1.

Corollary 5.11. *If $\Phi : \mathbb{R}_+ \to \mathbb{K}$ is a continuous function such that $\Phi(0) = \mathcal{E}$ and $\Phi(s) \leqslant \Phi(t)$ for $0 \leqslant s \leqslant t < \infty$, then $\Phi \equiv \mathcal{E}$.*

Proof. Suppose that Φ is a function with the stated properties. If $\Phi \not\equiv \mathcal{E}$, then there exist $0 < u < v < \infty$ such that $\Phi(u) < \Phi(v)$. It follows from Theorem 5.8 that there exists $\mathcal{Y} \in \mathbb{I}$ such that the multiplicity of $\mathcal{Y}$ in the factorization of $\Phi(v)$ is strictly greater than the multiplicity of $\mathcal{Y}$ in the factorization of $\Phi(u)$. Define $M : \mathbb{R}_+ \to \mathbb{N}$ by setting $M(s)$,

$s \geqslant 0$, to be the multiplicity of $\mathcal{Y}$ in the factorization of $\Phi(s)$. This function is nondecreasing and so there must exist $u \leqslant t \leqslant v$ such that $M(t-) < M(t+)$. It follows that $\Phi(t-\varepsilon) \boxplus \mathcal{Y} \boxplus \cdots \boxplus \mathcal{Y} \leqslant \Phi(t+\varepsilon)$ for all $\varepsilon > 0$, where there are $M(t+) - M(t-)$ summands in the sum, and this contradicts the continuity of Φ. □

The next result will be a consequence of the characterization of infinitely divisible random elements of $\mathbb{K}$ in Theorem 9.1, but we present it here as in illustration of a nonobvious and initially somewhat surprising feature of $\mathbb{K}$.

Corollary 5.12. *Suppose that $\mathcal{X} \in \mathbb{K}$ is infinitely divisible in the sense that for each positive integer n there exists $\mathcal{X}_n \in \mathbb{K}$ such that $\mathcal{X} = \mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$, where the sum has n terms. Then, $\mathcal{X} = \mathcal{E}$.*

Proof. This is immediate from Theorem 5.8. Indeed, a decomposition of the form $\mathcal{X} = \mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$ is only possible if n divides each of the multiplicities with which the various irreducible elements appear in the factorization of $\mathcal{X}$. □

Remark 5.13. A simpler and more direct proof of Corollary 5.12 is to note that if $\mathcal{X}$ can be written as $\mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$ for all n, then the push-forward of the probability measure $\mu_X^{\otimes 2}$ by the map $(x', x'') \to r_X(x', x'')$ is an infinitely divisible probability measure supported on $[0, \operatorname{diam}(\mathcal{X})]$ and hence it must be a point mass at zero, where we again use the fact that any infinitely divisible probability measure with bounded support is a point mass, a fact that, as we noted in Remark 5.3, has a simple direct proof. An argument along these lines can also be used to establish Corollary 5.11.

The following result is an immediate consequence of Corollary 5.12.

Corollary 5.14. *If $\Phi : \mathbb{R}_+ \to \mathbb{K}$ is a function such that $\Phi(0) = \mathcal{E}$ and $\Phi(s) \boxplus \Phi(t) = \Phi(s+t)$ for $0 \leqslant s, t < \infty$, then $\Phi \equiv \mathcal{E}$.*

Remark 5.15. Although Corollary 5.14 says there are no nontrivial *additive* functions from $\mathbb{R}_+$ to $\mathbb{K}$, there do exist nontrivial *superadditive functions*; that is, functions $\Phi : \mathbb{R}_+ \to \mathbb{K}$ such that $\Phi(0) = \mathcal{E}$ and $\Phi(s) \boxplus \Phi(t) \leqslant \Phi(s+t)$ for $0 \leqslant s, t < \infty$. For example, take $\mathcal{X} \in \mathbb{K} \backslash \{\mathcal{E}\}$ and set $\Phi(t) = \mathcal{X} \boxplus \cdots \boxplus \mathcal{X}$ for $n \leqslant t < n+1$, $n \in \mathbb{N}$, where the sum has n terms and we interpret the empty sum as $\mathcal{E}$. We have

$$\Phi(s) \boxplus \Phi(t) = \Phi(\lfloor s \rfloor) \boxplus \Phi(\lfloor t \rfloor) = \Phi(\lfloor s \rfloor + \lfloor t \rfloor) \leqslant \Phi(s+t).$$

However, by Corollary 5.11 there are no nontrivial continuous superadditive functions. Furthermore, there are no superadditive functions Φ such that $\Phi(t) \neq \mathcal{E}$ for all $t > 0$.

There are also nontrivial *subadditive functions*; that is, functions $\Phi : \mathbb{R}_+ \to \mathbb{K}$ such that $\Phi(0) = \mathcal{E}$ and $\Phi(s) \boxplus \Phi(t) \geqslant \Phi(s+t)$ for $0 \leqslant s, t < \infty$. For example, it suffices to take some $\mathcal{X} \in \mathbb{K}\backslash\{\mathcal{E}\}$ and set $\Phi(t) = \mathcal{X}$ for $t > 0$. However, there are no continuous subadditive functions because if Φ is such a function and $\mathcal{Y} \in \mathbb{I}$ is such that $\mathcal{Y} \leqslant \Phi(t)$, then it follows from $\Phi(\frac{t}{2}) \boxplus \Phi(\frac{t}{2}) \geqslant \Phi(t)$ that $\mathcal{Y} \leqslant \Phi(\frac{t}{2})$ and hence $\mathcal{Y} \leqslant \Phi(\frac{t}{2^n})$ for all $n \in \mathbb{N}$, but this contradicts the continuity of Φ at 0.

Remark 5.16. With Theorem 5.8 in hand, we can give a more concrete description of the group $\mathbb{G}$ described in Remark 4.3. Recall that $\mathcal{V}^{\boxplus n}$ denotes the sum of n terms $\mathcal{V} \boxplus \cdots \boxplus \mathcal{V}$, and we interpret the empty sum as $\mathcal{E}$. With this notation, any $\mathcal{U} \in \mathbb{K}$ has a unique representation as $\mathcal{U} = \boxplus_{\mathcal{V}\in\mathbb{I}} \mathcal{V}^{\boxplus n_{\mathcal{V}}}$, where $n_{\mathcal{V}} = 0$ for all but countably many $\mathcal{V} \in \mathbb{I}$ and $\sum_{\mathcal{V}\in\mathbb{I}} n_{\mathcal{V}} \operatorname{diam}(\mathcal{V}) < \infty$. Any element of $\mathbb{G}$ corresponds to a unique pair $(\boxplus_{\mathcal{V}\in\mathbb{I}} \mathcal{V}^{\boxplus n_{\mathcal{V}}^+}, \boxplus_{\mathcal{V}\in\mathbb{I}} \mathcal{V}^{\boxplus n_{\mathcal{V}}^-})$, where $n_{\mathcal{V}}^+ = 0$ and $n_{\mathcal{V}}^- = 0$ for all but countably many $\mathcal{V} \in \mathbb{I}$, $\sum_{\mathcal{V}\in\mathbb{I}} n_{\mathcal{V}}^+ \operatorname{diam}(\mathcal{V}) < \infty$ and $\sum_{\mathcal{V}\in\mathbb{I}} n_{\mathcal{V}}^- \operatorname{diam}(\mathcal{V}) < \infty$, and $n_{\mathcal{V}}^+ n_{\mathcal{V}}^- = 0$ for all $\mathcal{V} \in \mathbb{I}$. We can therefore identify an element of $\mathbb{G}$ with the corresponding object $((n_{\mathcal{V}}^+, n_{\mathcal{V}}^-))_{\mathcal{V}\in\mathbb{I}}$. In terms of this representation, the binary operation on $\mathbb{G}$ transforms the two objects $((m_{\mathcal{V}}^+, m_{\mathcal{V}}^-))_{\mathcal{V}\in\mathbb{I}}$ and $((n_{\mathcal{V}}^+, n_{\mathcal{V}}^-))_{\mathcal{V}\in\mathbb{I}}$ into the object

$$((m_{\mathcal{V}}^+ + n_{\mathcal{V}}^+ - [m_{\mathcal{V}}^+ + n_{\mathcal{V}}^+] \wedge [m_{\mathcal{V}}^- + n_{\mathcal{V}}^-], m_{\mathcal{V}}^- + n_{\mathcal{V}}^- - [m_{\mathcal{V}}^+ + n_{\mathcal{V}}^+] \wedge [m_{\mathcal{V}}^- + n_{\mathcal{V}}^-]))_{\mathcal{V}\in\mathbb{I}}.$$

6. Prime factorizations as measures

Theorem 5.8 guarantees that any $\mathcal{X} \in \mathbb{K}$ has a unique representation as $\mathcal{X} = \boxplus_k \mathcal{Y}_k^{\boxplus m_k}$, where the $\mathcal{Y}_k \in \mathbb{I}$ are distinct, the integers m_k are positive, and we define the empty sum to be $\mathcal{E}$. The number of $\mathcal{Y}_k$ outside any neighborhood of $\mathcal{E}$ is finite. It is natural to code such a factorization as the measure $\Psi(\mathcal{X}) := \sum_k m_k \delta_{\mathcal{Y}_k}$ on $\mathbb{K}$ that is concentrated on $\mathbb{I}$ and assigns mass m_k to the point $\mathcal{Y}_k$ for each k.

Denote by $\mathfrak{N}$ the family of Borel measures N on $\mathbb{K}$ such that $N(\mathbb{K}\backslash\mathbb{I}) = 0$ and $N(B) \in \mathbb{N}$ for every Borel set B that does not intersect some neighborhood of $\mathcal{E}$. Any $N \in \mathfrak{N}$ can be represented as the positive integer linear combination of Dirac measures

$$N = \sum_k m_k \delta_{\mathcal{Y}_k}$$

for distinct $\mathcal{Y}_k \in \mathbb{I}$ and positive integers m_k, where the sum may be finite or countably infinite depending on the cardinality of the support of N. Given $N \in \mathfrak{N}$ with such a representation we define a unique element of $\mathbb{K}$ by

$$\Sigma(N) := \boxplus_k \mathcal{Y}_k^{\boxplus m_k},$$

if the sum converges (recall from Corollary 3.7 that the convergence of the sum is independent of the order summands). Thus, $\Sigma(\Psi(\mathcal{X})) = \mathcal{X}$ for all $\mathcal{X} \in \mathbb{K}$.

It is possible to topologize $\mathfrak{N}$ with the metrizable $w^{\#}$-topology of [DVJ03, Section A2.6]. This topology is the topology generated by integration against bounded continuous functions that are supported outside a neighborhood of $\mathcal{E}$. The resulting Borel σ-field coincides σ-field generated by the $\mathbb{N}$-valued maps $N \mapsto N(B)$ Borel measurable, where B is a Borel subset of $\mathbb{K}$ that is disjoint from some neighborhood of $\mathcal{E}$, see [DVJ03, Theorem A2.6.III].

Proposition 6.1. *The map $\Psi : \mathbb{K} \to \mathfrak{N}$ is Borel measurable.*

Proof. The set $\{(\mathcal{X}, \mathcal{Y}) \in \mathbb{K}^2 : \mathcal{Y} \leqslant \mathcal{X}\}$ is closed by part (a) of Corollary 4.4 and the set $\mathbb{I}$ is a G_δ by Proposition 5.1. It follows that the set $\mathbb{B} := \{(\mathcal{X}, \mathcal{Y}) \in \mathbb{K}^2 : \mathcal{Y} \leqslant \mathcal{X}, \mathcal{Y} \in \mathbb{I}\}$ is a G_δ subset of $\mathbb{K}^2$ and, in particular, it is Borel.

For any $\mathcal{X} \in \mathbb{K}$, the section $\mathbb{B}_{\mathcal{X}} := \{\mathcal{Y} \in \mathbb{K} : (\mathcal{X}, \mathcal{Y}) \in \mathbb{B}\} = \{\mathcal{Y} \in \mathbb{K} : \mathcal{Y} \leqslant \mathcal{X}, \mathcal{Y} \in \mathbb{I}\}$ is countable (indeed, it is discrete with $\mathcal{E}$ as its only possible accumulation point).

By [Kec95, Exercise 18.15], the sets $\mathbb{M}_n := \{\mathcal{X} \in \mathbb{K} : \#\mathbb{B}_{\mathcal{X}} = n\}$, $n = 1, 2, \ldots, \infty$, are Borel and for each n there exist Borel functions $(\theta_i^{(n)})_{0 \leqslant i < n}$ such that:

- $\theta_i^{(n)} : \mathbb{M}_n \to \mathbb{K}$,
- the sets $\{(\mathcal{X}, \mathcal{Y}) : \mathcal{X} \in \mathbb{M}_n, \mathcal{Y} = \theta_i^{(n)}(\mathcal{X})\}$, $0 \leqslant i < n$, $n = 1, 2, \ldots, \infty$, are pairwise disjoint,
- $\mathbb{B}_{\mathcal{X}} = \{\theta_i^{(n)}(\mathcal{X}) : 0 \leqslant i < n\}$ for $\mathcal{X} \in \mathbb{M}_n$, $n = 1, 2, \ldots, \infty$.

Recall the Borel function M from part (b) of Corollary 4.4. For $\mathcal{X} \in \mathbb{M}_n$, the set $\{(\theta_i^{(n)}(\mathcal{X}), M(\mathcal{X}, \theta_i^{(n)}(\mathcal{X})) : 0 \leqslant i < n\}$ is a listing of the elements of the set $\{\mathcal{Y} \in \mathbb{I} : \mathcal{Y} \leqslant \mathcal{X}\}$ along with their multiplicities in the prime factorization of $\mathcal{X}$. The functions $\mathcal{X} \mapsto (\theta_i^{(n)}(\mathcal{X}), M(\mathcal{X}, \theta_i^{(n)}(\mathcal{X}))$, $\mathcal{X} \in \mathbb{M}_n$, $0 \leqslant i < n$, $n = 1, 2, \ldots, \infty$, are measurable and so

$$\mathcal{X} \mapsto \Psi(\mathcal{X}) = \sum_{i=0}^{n} M(\mathcal{X}, \theta_i^{(n)}(\mathcal{X})) \delta_{\theta_i^{(n)}(\mathcal{X})}$$

for $\mathcal{X} \in \mathbb{M}_n$, provides a measurable map from $\mathbb{K}$ to $\mathfrak{N}$, see [DVJ08, Proposition 9.1.X]. □

Remark 6.2. The map Ψ is not continuous for the $w^{\#}$-topology. In fact, any $\mathcal{X} \in (\mathbb{K} \backslash \mathbb{I}) \backslash \{\mathcal{E}\}$ is a discontinuity point, as the following argument demonstrates. Because $\mathbb{I}$ of is dense in $\mathcal{K}$, it is possible to find a sequence $\mathcal{X}_n \in \mathbb{I}$ that converges to $\mathcal{X}$. Therefore, $\Psi(\mathcal{X}_n) = \delta_{\mathcal{X}_n}$, whereas

$\Psi(\mathcal{X})$ has total mass at least two and the distance between any atom of $\Psi(\mathcal{X})$ and the point $\mathcal{X}_n$ is bounded away from zero uniformly in n.

We omit the straightforward proof of the following result.

Lemma 6.3. *The set $\{N \in \mathfrak{N} : \Sigma(N)$ is defined$\}$ is measurable and the restriction of the map Σ to this set is measurable.*

7. Scaling

Given $\mathcal{X} \in \mathbb{K}$ and $a > 0$, set $a\mathcal{X} := (X, ar_X, \mu_X) \in \mathbb{K}$. This *scaling* operation operation is continuous and satisfies the first distributivity law

$$a(\mathcal{X} \boxplus \mathcal{Y}) = (a\mathcal{X}) \boxplus (a\mathcal{Y}) \quad \text{for } \mathcal{X}, \mathcal{Y} \in \mathbb{K} \text{ and } a > 0. \tag{7.1}$$

The semigroup $(\mathbb{K}, \boxplus)$ equipped with this scaling operation is a convex cone. The neutral element $\mathcal{E}$ is the origin in this cone; that is, $\lim_{a\downarrow 0} a\mathcal{X} = \mathcal{E}$ for all $\mathcal{X} \in \mathbb{K}$. Note that $\operatorname{diam}(a\mathcal{X}) = a \operatorname{diam}(\mathcal{X})$ for $\mathcal{X} \in \mathbb{K}$ and $a > 0$. While the Gromov–Prohorov metric is not homogeneous for the scaling operation, the $\mathbb{D}$-metric introduced in [Stu06] is homogeneous and yields the same topology on $\mathbb{K}$.

It follows from (7.1) that $\mathcal{Y} \in \mathbb{I}$ if and only if $a\mathcal{Y} \in \mathbb{I}$ for all $a > 0$.

There is an analogue of the scaling operation for the semigroup of semicharacters $(\chi_A)_{A\in\mathbb{A}}$ given by

$$a\chi_A(\mathcal{X}) := \chi_A(a\mathcal{X}) = \chi_{aA}(\mathcal{X}), \quad a > 0,\ A \in \mathbb{A},\ \mathcal{X} \in \mathbb{K}.$$

We have seen that $(\mathbb{K}, \leqslant)$ is a distributive lattice. There is a large literature on lattices that are equipped with an action of the additive group of the real numbers (see, for example [Kap48, Pie59, Hol69]). Using exponential and logarithms to go back and forth from one setting to the other, this work can be recast as being about lattices with an action of the group consisting of $\mathbb{R}_{++} = (0, \infty)$ equipped with the usual multiplication of real numbers. Unfortunately, one of the hypotheses usually assumed in this area translates to our setting as an assumption that $\mathcal{X} < a\mathcal{X}$ for $a > 1$. The following result shows that this is far from being the case and also that scaling operation certainly does not satisfy the second distributivity law.

Proposition 7.1. a) *If $\mathcal{X} \leqslant a\mathcal{X}$ for some $\mathcal{X} \in \mathbb{K}$ and $a \neq 1$, then $a > 1$ and $\mathcal{X} = \boxplus_{k=1}^{\infty} a^{-k}\mathcal{Z}$, where $\mathcal{Z}$ is defined by the requirement that $a\mathcal{X} = \mathcal{X} \boxplus \mathcal{Z}$.*

b) *If $(a\mathcal{X}) \boxplus (b\mathcal{X}) = c\mathcal{X}$, for some $\mathcal{X} \in \mathbb{K}$ and $a, b, c > 0$, then $\mathcal{X} = \mathcal{E}$.*

Proof. Consider part (a). Suppose that $\mathcal{X} \neq \mathcal{E}$ is such that $\mathcal{X} \leqslant a\mathcal{X}$ for $a \neq 1$. Because $\operatorname{diam}(\mathcal{X}) \leqslant \operatorname{diam}(a\mathcal{X}) = a \operatorname{diam}(\mathcal{X})$, it must be the case that $a > 1$. It follows from Proposition 4.2 that $\mathcal{Z}$ well-defined. We have $\mathcal{X} = a^{-1}\mathcal{Z} \boxplus a^{-1}\mathcal{X}$. Iterating, we have $\mathcal{X} = \boxplus_{k=1}^{n} a^{-k}\mathcal{Z} \boxplus a^{-n}\mathcal{X}$. By part (a) of Corollary 3.6 (or part (b) of Corollary 3.6), $\boxplus_{k=1}^{n} a^{-k}\mathcal{Z}$ exists. Moreover, $\lim_{n\to\infty} \operatorname{diam}(a^{-n}\mathcal{X}) = 0$, and part (a) follows.

Consider part (b). Suppose that $(a\mathcal{X}) \boxplus (b\mathcal{X}) = c\mathcal{X}$ for some $\mathcal{X} \in \mathbb{K}$ and $a, b, c > 0$. By part (a) of Lemma 2.4, $(a + b) \operatorname{diam}(\mathcal{X}) = \operatorname{diam}((a\mathcal{X}) \boxplus (b\mathcal{X})) = \operatorname{diam}(c\mathcal{X}) = c \operatorname{diam}(\mathcal{X})$, and so $a + b = c$. An irreducible element $\mathcal{Y} \in \mathbb{I}$ appears in the factorization of $\mathcal{X}$ guaranteed by Theorem 5.8 if and only if $c\mathcal{Y} \in \mathbb{I}$ appears in the factorization of $c\mathcal{X}$, and similar remarks hold for the factorizations of $a\mathcal{X}$ and $b\mathcal{X}$. Suppose that δ is the maximum of the diameters of the irreducible elements that appear in the factorization of $\mathcal{X}$. (There could, of course, be more than one - but only finitely many - irreducible factors with maximal diameter.) It follows that $c\mathcal{X}$ has in irreducible factor with diameter $c\delta$, whereas all the irreducible factors of $(a\mathcal{X}) \boxplus (b\mathcal{X})$ have diameters at most $(a \vee b)\delta$, which is impossible since $(a \vee b) < c$. □

Remark 7.2. While it is possible to introduce a notion of convexity for subsets of $\mathbb{K}$ using the addition and scaling in an obvious way, the absence of the second distributivity law makes the situation entirely different from the classical case. For instance, a single point $\{\mathcal{X}\}$ is not convex for $\mathcal{X} \neq \mathcal{E}$ and its convex hull is the set of spaces of the form $a_1\mathcal{X} \boxplus \cdots \boxplus a_n\mathcal{X}$ for $a_1, \ldots, a_n \geqslant 0$ such that $a_1 + \cdots + a_n = 1$. It is a consequence of Remark 8.3 that this latter set is not even pre-compact.

Remark 7.3. The map that sends $a \in \mathbb{R}_{++}$ to the automorphism $\mathcal{X} \mapsto a\mathcal{X}$ of $(\mathbb{K}, \boxplus)$ is a homomorphism from $(\mathbb{R}_{++}, \times)$ to the group of automorphisms of $(\mathbb{K}, \boxplus)$. We can therefore define the *semidirect product* $\mathbb{K} \rtimes \mathbb{R}_{++}$ to be the semigroup consisting of the set $\mathbb{K} \times \mathbb{R}_{++}$ equipped with the operation $\circledast$ defined by

$$(\mathcal{X}, a) \circledast (\mathcal{Y}, b) := (\mathcal{X} \boxplus (a\mathcal{Y}), ab).$$

This semigroup has the identity element $(\mathcal{E}, 1)$ and is noncommutative. The semidirect product of the group $(\mathbb{G}, \boxplus)$ considered in Remark 4.3 and Remark 5.16 and the group $(\mathbb{R}_{++}, \times)$ can be defined similarly. It would be interesting to extend the investigation of infinite divisibility in Section 9 to this semigroup and group, but we leave this topic for future study.

8. The Laplace transform

A random element in $\mathbb{K}$ is defined with respect to the Borel σ-algebra on $\mathbb{K}$ generated by the Gromov–Prohorov metric.

Lemma 8.1. *Two $\mathbb{K}$-valued random elements $\mathbf{X}$ and $\mathbf{Y}$ have the same distribution if and only if $\mathbb{E}[\chi_A(\mathbf{X})] = \mathbb{E}[\chi_A(\mathbf{Y})]$ for all $A \in \mathbb{A}$.*

Proof. It follows from Lemma 3.5 that the set of functions $\{\chi_A : A \in \mathbb{A}\}$ generates the Borel σ-algebra on $\mathbb{K}$. From Remark 3.3, this set is a semigroup under the usual multiplication of functions and, in particular, it is closed under multiplication. The result now follows from a standard monotone class argument. □

Remark 8.2. Recall from Section 6 the set $\mathfrak{N}$ of $\mathbb{N}$-valued measures that are concentrated on $\mathbb{I}$ and the associated measurable structure. Following the usual terminology, we define a *point process* to be a random element of $\mathfrak{N}$. It follows from Proposition 6.1 that any $\mathbb{K}$-valued random element $\mathbf{X}$ can, in the notation of Section 6, be viewed as a point process $\mathbf{N} := \Psi(\mathbf{X})$ such that $\Sigma(\mathbf{N}) = \mathbf{X}$. If we write $\mathbf{N} = \sum m_k \delta_{\mathbf{Y}_k}$ on $\mathbb{I}$, then

$$\mathbb{E}[\chi_A(\mathbf{X})] = \mathbb{E}[\chi_A(\Sigma(\Psi(\mathbf{X}))] = \mathbb{E}\left[\prod \chi_A(\mathbf{Y}_k)^{m_k}\right].$$

The right-hand side is the expected value of the product of the function χ_A applied to each of the atoms of $\mathbf{N}$ taking into account their multiplicities and hence it is an instance of the *probability generating functional* of the point process $\mathbf{N}$, see [DVJ08, Equation (9.4.13)].

Remark 8.3. A fairly immediate consequence of Lemma 8.1 is that there is no analogue of a law of large numbers for random elements of $\mathbb{K}$ in the sense that if $(\mathbf{X}_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of random elements of $\mathbb{K}$ that are not identically equal $\mathcal{E}$, then $\frac{1}{n}\boxplus_{k=0}^{n-1}\mathbf{X}_k$ does not even have a subsequence that converges in distribution. Indeed, for $A \in \mathbb{A}$

with $A \in \mathbb{R}_+^{\binom{m}{2}}$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\chi_A\left(\frac{1}{n}\boxplus_{k=0}^{n-1}\mathbf{X}_k\right)\right] \\
&\quad = \lim_{n\to\infty}\left(\mathbb{E}\left[\chi_A\left(\frac{1}{n}\mathbf{X}_1\right)\right]\right)^n \\
&\quad = \lim_{n\to\infty}\left(\int_{\mathbb{K}}\int_{X^m}\exp\left(-\frac{1}{n}\sum_{1\leqslant i<j\leqslant m} a_{ij}r_X(x_i,x_j)\right)\right. \\
&\qquad \left. \times\ \mu_X^{\otimes m}(dx)\,\mathbb{P}\{\mathbf{X}_1\in d\mathcal{X}\}\right)^n \\
&\quad = \exp\left(-\int_{\mathbb{K}}\int_{X^m}\sum_{1\leqslant i<j\leqslant m} a_{ij}r_X(x_i,x_j)\,\mu_X^{\otimes m}(dx)\,\mathbb{P}\{\mathbf{X}_1\in d\mathcal{X}\}\right) \\
&\quad = \exp\left(-\sum_{1\leqslant i<j\leqslant m} a_{ij}\int_{\mathbb{K}}\int_{X^2} r_X(x_1,x_2)\,\mu_X^{\otimes 2}(dx)\,\mathbb{P}\{\mathbf{X}_1\in d\mathcal{X}\}\right)
\end{aligned}$$

by a standard argument that is used to prove the weak law of large numbers for a sequence of i.i.d. nonnegative random variables using Laplace transforms. If some subsequence of $\frac{1}{n}\boxplus_{k=0}^{n-1}\mathbf{X}_k$ converged in distribution to a limit $\mathbf{Y}$, then we would have

$$\begin{aligned}
&\int_{\mathbb{K}}\int_{Y^m}\exp\left(-\sum_{1\leqslant i<j\leqslant m} a_{ij}r_Y(y_i,y_j)\right)\mu_Y^{\otimes m}(dx)\,\mathbb{P}\{\mathbf{Y}\in d\mathcal{Y}\} \\
&\quad = \exp\left(-\sum_{1\leqslant i<j\leqslant m} a_{ij}\int_{\mathbb{K}}\int_{X^2} r_X(x_1,x_2)\,\mu_X^{\otimes 2}(dx)\,\mathbb{P}\{\mathbf{X}_1\in d\mathcal{X}\}\right).
\end{aligned}$$

By the unicity of Laplace transforms for nonnegative random vectors, this implies that

$$\begin{aligned}
&\int_{\mathbb{K}}\mu_Y^{\otimes 2}\Bigg\{(y_1,y_2)\in Y^2 : r_Y(y_1,y_2) \\
&\qquad \neq \int_{X^2} r_X(x_1,x_2)\,\mu_X^{\otimes 2}(dx)\,\mathbb{P}\{\mathbf{X}_1\in d\mathcal{X}\}\Bigg\}\,\mathbb{P}\{\mathbf{Y}\in d\mathcal{Y}\} \\
&\quad = 0,
\end{aligned}$$

and hence there is a constant $c > 0$ such that for $\mathbb{P}\{\mathbf{Y}\in\cdot\}$-almost all $Y\in\mathbb{K}$ we have $r_Y(y_1,y_2) = c$ for $\mu_Y^{\otimes 2}$-almost all $(y_1,y_2)\in Y^2$, but this is impossible for nontrivial a compact metric space (Y,r_Y) and probability measure μ_Y with full support.

9. Infinitely divisible random elements

A random element $\mathbf{Y}$ of $\mathbb{K}$ is *infinitely divisible* if for each positive integer n there are i.i.d. random elements $\mathbf{Y}_{n1}, \dots, \mathbf{Y}_{nn}$ such that $\mathbf{Y}$ has the same distribution as $\mathbf{Y}_{n1} \boxplus \cdots \boxplus \mathbf{Y}_{nn}$.

A $\mathbb{K}$-valued *Lévy process* is a $\mathbb{K}$-valued stochastic process $(\mathbf{X}_t)_{t\geqslant 0}$ such that:

- $\mathbf{X}_0 = \mathcal{E}$
- $t \mapsto \mathbf{X}_t$ is càdlàg (that is, right-continuous with left-limits)
- Given $0 = t_0 < t_1 < \ldots < t_n$, there are independent $\mathbb{K}$-valued random variables $\mathbf{Z}_{t_0 t_1}, \mathbf{Z}_{t_1 t_2}, \dots, \mathbf{Z}_{t_{n-1} t_n}$ such that the distribution of $\mathbf{Z}_{t_m t_{m+1}}$ only depends on $t_{m+1} - t_m$ for $0 \leqslant m \leqslant n-1$ and $\mathbf{X}_{t_\ell} = \mathbf{X}_{t_k} \boxplus \mathbf{Z}_{t_k t_{k+1}} \boxplus \cdots \boxplus \mathbf{Z}_{t_{\ell-1} t_\ell}$ for $0 \leqslant k < \ell \leqslant n$.

An account of the general theory of infinitely divisible distributions on commutative semigroups may be found in [BCR84]. The following result is the analogue in our setting of the classical Lévy-Hinčin-Itô description of an infinitely divisible, real-valued random variable.

Theorem 9.1. a) *A random element $\mathbf{Y}$ of $\mathbb{K}$ is infinitely divisible if and only if it has the same distribution as $\mathbf{X}_1$, where $(\mathbf{X}_t)_{t\geqslant 0}$ is a Lévy process with distribution uniquely specified by that of $\mathbf{Y}$.*

b) *For each $t > 0$ there is a unique random element $\Delta\mathbf{X}_t$ such that $\mathbf{X}_t = \mathbf{X}_{t-} \boxplus \Delta\mathbf{X}_t$.*

c) *For each $t > 0$, $\mathbf{X}_t = \boxplus_{0<s\leqslant t} \Delta\mathbf{X}_s$, where the sum is a well-defined limit that does not depend on the order of the summands.*

d) *The set of points $\{(t, \Delta\mathbf{X}_t) : \Delta\mathbf{X}_t \neq \mathcal{E}\}$ form a Poisson point process on $\mathbb{R}_+ \times (\mathbb{K}\backslash\{\mathcal{E}\})$ with intensity measure $\lambda \otimes \nu$, where λ is Lebesgue measure and ν is a σ-finite measure on $\mathbb{K}\backslash\{\mathcal{E}\}$ such that*

$$\int (\operatorname{diam}(\mathcal{X}) \wedge 1)\, \nu(d\mathcal{X}) < \infty\,. \tag{9.1}$$

e) *Conversely, if ν is a σ-finite measure on $\mathbb{K}\backslash\{\mathcal{E}\}$ satisfying* (9.1), *then there is an infinitely divisible random element $\mathbf{Y}$ and a Lévy process $(\mathbf{X}_t)_{t\geqslant 0}$ such that (a)-(d) hold, and the distributions of this random element and Lévy process are unique.*

Proof. Write $\mathbb{D}$ for the set of nonnegative dyadic rational numbers. It follows from the infinite divisibility of $\mathbf{Y}$ and the Kolmogorov extension theorem that we can build a family of random variables $(\mathbf{X}_q)_{q\in\mathbb{D}}$ such that:

- $\mathbf{X}_0 = \mathcal{E}$,

- $\mathbf{X}_1$ has the same distribution as $\mathbf{Y}$,
- Given $q_0, \ldots, q_n \in \mathbb{D}$ with $0 = q_0 < q_1 < \ldots < q_n$, there are independent $\mathbb{K}$-valued random variables $\mathbf{Z}_{q_0 q_1}, \mathbf{Z}_{q_1 q_2}, \ldots, \mathbf{Z}_{q_{n-1} q_n}$ such that the distribution of $\mathbf{Z}_{q_m q_{m+1}}$ only depends on $q_{m+1} - q_m$ for $0 \leqslant m \leqslant n-1$ and $\mathbf{X}_{q_\ell} = \mathbf{X}_{q_k} \boxplus \mathbf{Z}_{q_k q_{k+1}} \boxplus \cdots \boxplus \mathbf{Z}_{q_{\ell-1} q_\ell}$ for $0 \leqslant k < \ell \leqslant n$. In particular, $\mathbf{X}_p \leqslant \mathbf{X}_q$ for $p, q \in \mathbb{D}$ with $p \leqslant q$.

We claim that if $p \in \mathbb{D}$, then

$$\lim_{q \downarrow p,\, q \in \mathbb{D}} \mathbf{X}_q = \mathbf{X}_p, \quad \text{a.s.} \tag{9.2}$$

To see that this is the case, note that if $p, q \in \mathbb{D}$ with $p < q$, then $\mathbf{X}_q = \mathbf{X}_p \boxplus \mathbf{Z}_{pq}$ and it suffices to show that $\lim_{q \downarrow p,\, q \in \mathbb{D}} d_{\mathrm{GPr}}(\mathbf{Z}_{pq}, \mathcal{E}) = 0$ almost surely.

By part (b) of Lemma 2.4, it will certainly suffice to show that $\lim_{q \downarrow p,\, q \in \mathbb{D}} \operatorname{diam}(\mathbf{Z}_{pq}) = 0$ a.s. However, note that if we set $T_0 = 0$ and $T_r = \operatorname{diam}(\mathbf{Z}_{p,p+r})$ for $r \in \mathbb{D} \backslash \{0\}$, then the $\mathbb{R}_+$-valued process $(T_r)_{r \in \mathbb{D}}$ has stationary independent increments. It is well-known that such a process has a càdlàg extension to the index set $\mathbb{R}_+$ and hence, in particular, $\lim_{r \downarrow 0,\, r \in \mathbb{D}} T_r = 0$.

Lemma 9.4 applied to $(\mathbf{X}_p)_{p \in \mathbb{D}}$ gives that it is possible to extend $(\mathbf{X}_p)_{p \in \mathbb{D}}$ to a Lévy process $(\mathbf{X}_t)_{t \geqslant 0}$. This establishes (a). Moreover, for each $t > 0$ there is a unique $\mathbb{K}$-valued random variable $\Delta \mathbf{X}_t$ such that $\mathbf{X}_t = \mathbf{X}_{t-} \boxplus \Delta \mathbf{X}_t$, $\sum_{0 < s \leqslant t} \operatorname{diam}(\Delta \mathbf{X}_s)$ is finite, and $\mathbf{X}_t = \boxplus_{0 < s \leqslant t} \Delta \mathbf{X}_s$, where the sum is well-defined. This establishes (b) and (c).

A standard argument (see, for example, [Kal02, Theorem 12.10]) shows that the set of points $\{(t, \Delta \mathbf{X}_t) : \Delta \mathbf{X}_t \neq \mathcal{E}\}$ form a Poisson point process on $\mathbb{R}_+ \times (\mathbb{K} \backslash \{\mathcal{E}\})$. The stationarity of the "increments" of $(\mathbf{X}_t)_{t \geqslant 0}$ forces the intensity measure of this Poisson point process to be of the form $\lambda \otimes \nu$, and the fact that $\sum_{0 < s \leqslant t} \operatorname{diam}(\Delta \mathbf{X}_s)$ is finite for all $t \geqslant 0$ implies (9.1), see, for example, [Kal02, Corollary 12.11]. This establishes (d).

We omit the straightforward proof of (e). □

Following the usual terminology, we refer to the σ-finite measure in Theorem 9.1 as the Lévy measure of the infinitely divisible random element $\mathbf{Y}$ or the Lévy process $(\mathbf{X}_t)_{t \geqslant 0}$. The following is immediate from Theorem 9.1, the multiplicative property of the semicharacters χ_A, and the usual formula for the Laplace functional of a Poisson process.

Corollary 9.2. *If $\mathbf{Y}$ is an infinitely divisible random element of $\mathbb{K}$ with Lévy measure ν, then the Laplace transform of $\mathbf{Y}$ is given by*

$$\mathbb{E}[\chi_A(\mathbf{Y})] = \exp\left(- \int (1 - \chi_A(\mathcal{Y}))\, \nu(d\mathcal{Y}) \right), \quad A \in \mathbb{A}. \tag{9.3}$$

Remark 9.3. In the notation of Theorem 9.1, the random measure

$$\sum_{0<t\leqslant 1} \delta_{\Delta \mathbf{X}_t}$$

is a Poisson random measure on $\mathbb{K}$ with intensity measure ν and we have $\mathbf{Y} = \mathbf{X}_1 = \boxplus_{0<t\leqslant 1} \Delta \mathbf{X}_t$. The push-forward of this random measure by the map Ψ of Proposition 6.1 is a Poisson random measure on the space $\mathfrak{N}$ of $\mathbb{N}$-valued measures that are concentrated on $\mathbb{I}$. The intensity measure of this latter Poisson random measure is the push-forward Q of the Lévy measure ν by Ψ. The "points" of the latter Poisson random measure are usually called *clusters* in the point processes literature, while Q itself is called the *KLM measure*, see [DVJ08, Definition 10.2.IV]. Let $\mathbf{N}$ be the point process on $\mathbb{I}$ obtained as the superposition of clusters; that is, $\mathbf{N} = \sum_{0<t\leqslant 1} \Psi(\Delta \mathbf{X}_t)$ is the sum of the $\mathbb{N}$-valued measures given by each individual cluster. This point process on $\mathbb{I}$ is called the *Poisson cluster process* in the Poisson point process literature. The infinite divisibility of $\mathbf{Y}$ implies the infinite divisibility of the point process $\Psi(\mathbf{Y})$ and the equality $\Psi(\mathbf{Y}) = \mathbf{N}$ is an instance of the well-known fact that infinitely divisible point processes are Poisson cluster processes. Furthermore, (9.3) corresponds to the classical representation of the probability generating functional of an infinitely divisible point process specialized to the space $\mathbb{I}$, see [DVJ08, Theorem 10.2.V]. On the other hand, if $\mathbf{M}$ is a Poisson cluster process on $\mathbb{I}$ such that $\Sigma(\mathbf{M})$ is almost surely well-defined, then $\Sigma(\mathbf{M})$ is an infinitely divisible random element of $\mathbb{K}$, and our observations above show that all infinitely divisible random elements of $\mathbb{K}$ appear this way.

We end this section with a deterministic path-regularization result that was used in the proof of Theorem 9.1.

Lemma 9.4. *Suppose that $\Xi : \mathbb{D} \to \mathbb{K}$ is such that $\Xi(0) = \mathcal{E}$, $\Xi(p) \leqslant \Xi(q)$ for $p, q \in \mathbb{D}$ with $0 \leqslant p \leqslant q$, and $\lim_{q\downarrow p,\, q\in\mathbb{D}} \Xi(q) = \Xi(p)$ for all $p \in \mathbb{D}$. Then, $\bar{\Xi}(t) := \lim_{q\downarrow t,\, q\in\mathbb{D}} \Xi(q)$ exists for all $t \in \mathbb{R}_+$. Moreover, the function $\bar{\Xi} : \mathbb{R}_+ \to \mathbb{K}$ has the following properties:*

- *$\bar{\Xi}(p) = \Xi(p)$ for $p \in \mathbb{D}$,*
- *$\bar{\Xi}(s) \leqslant \bar{\Xi}(t)$ for $s, t \in \mathbb{R}_+$ with $s \leqslant t$,*
- *$t \mapsto \bar{\Xi}(t)$ is càdlàg,*
- *for $p, q \in \mathbb{D}$ with $0 \leqslant p < q$, there is a unique $\Theta(p, q) \in \mathbb{K}$ such that $\Xi(q) = \Xi(p) \boxplus \Theta(p, q)$,*
- *for $0 \leqslant s < t$, there is a unique $\bar{\Theta}(s, t) \in \mathbb{K}$ such that $\bar{\Xi}(t) = \bar{\Xi}(s) \boxplus \bar{\Theta}(s, t)$ and $\bar{\Theta}(s, t) = \lim_{p\downarrow s,\, q\downarrow t,\, p,q\in\mathbb{D}} \Theta(p, q)$,*
- *for each $t > 0$ there is a unique $\Delta\bar{\Xi}(t) \in \mathbb{K}$ such that $\bar{\Xi}(t) = \lim_{s\uparrow t} \bar{\Xi}(s) \boxplus \Delta\bar{\Xi}(t)$*

- $\sum_{u<t\leqslant v} \operatorname{diam}(\Delta\bar{\Xi}(t)) \leqslant \operatorname{diam}(\bar{\Theta}(u,v))$ *for all* $0 \leqslant u < v$,
- *the sum* $\boxplus_{0<s\leqslant t} \Delta\bar{\Xi}(s)$ *is well-defined for all* $t \geqslant 0$,
- $\bar{\Xi}(t) = \boxplus_{0<s\leqslant t} \Delta\bar{\Xi}(s)$ *for all* $t \geqslant 0$.

Proof. It follows from part (b) of Corollary 3.6 that $\lim_{q\downarrow t,\, q\in\mathbb{D}} \Xi(q) =: \bar{\Xi}(t)$ exists for all $t \geqslant 0$.

It is clear that $\bar{\Xi}(p) = \Xi(p)$ for $p \in \mathbb{D}$ and that $\bar{\Xi}(s) \leqslant \bar{\Xi}(t)$ for $s,t \in \mathbb{R}_+$ with $s \leqslant t$. It is also clear that $t \mapsto \bar{\Xi}(t)$ is right-continuous. It follows from part (b) of Corollary 3.6 that $\bar{\Xi}(t-) := \lim_{s\uparrow t} \bar{\Xi}(s)$ exists for all $t > 0$ and $\bar{\Xi}(t-) \leqslant \bar{\Xi}(t)$ for all $t > 0$.

The existence and uniqueness of $\bar{\Theta}(s,t)$ such that $\bar{\Xi}(t) = \bar{\Xi}(s) \boxplus \bar{\Theta}(s,t)$ and the fact that $\bar{\Theta}(s,t) = \lim_{p\downarrow s,\, q\downarrow t,\, p,q\in\mathbb{D}} \Theta(p,q)$ follow from Proposition 4.2.

It also follows from Proposition 4.2 that $\Delta\bar{\Xi}(t)$ exists and is well-defined.

For any $0 \leqslant u < v$ and $u < t_1 < \cdots < t_n \leqslant v$ we have $\Delta\bar{\Xi}(t_1) \boxplus \cdots \boxplus \Delta\bar{\Xi}(t_n) \leqslant \bar{\Theta}(u,v)$. It follows from part (a) of Corollary 3.7 that $\boxplus_{0<s\leqslant t} \Delta\bar{\Xi}(s)$ is well-defined.

It is clear that $\boxplus_{0<s\leqslant t} \Delta\bar{\Xi}(s) \leqslant \bar{\Xi}(t)$ for all $t \geqslant 0$ and so we can use Proposition 4.2 to define a unique function $\Phi : \mathbb{R}_+ \to \mathbb{K}$ such that $\bar{\Xi}(t) = \Phi(t) \boxplus \boxplus_{0<s\leqslant t} \Delta\bar{\Xi}(s)$ for all $t \geqslant 0$. The function Φ is continuous and $\Phi(s) \leqslant \Phi(t)$ for $0 \leqslant s < t$. Also, $\Phi(0) = \mathbb{E}$. It follows from Corollary 5.11 that $\Phi \equiv \mathcal{E}$, completing the proof of the lemma. $\square$

10. Stable random elements

A $\mathbb{K}$-valued random element $\mathbf{Y}$ is *stable* with index $\alpha > 0$ if for any $a, b > 0$ the random element $(a+b)^{\frac{1}{\alpha}}\mathbf{Y}$ has the same distribution as $a^{\frac{1}{\alpha}}\mathbf{Y}' \boxplus b^{\frac{1}{\alpha}}\mathbf{Y}''$, where $\mathbf{Y}'$ and $\mathbf{Y}''$ are independent copies of $\mathbf{Y}$. Note that a stable random element is necessarily infinitely divisible. If $\mathbf{Y}$ is stable, then its diameter is a nonnegative strictly stable random variable.

There is a general investigation of stable random elements of convex cones in [DMZ08]. In general, not all such objects have Laplace transforms that are of the type analogous to those described in Corollary 9.2. For example, there can be Gaussian-like distributions. However, no such complexities arise in our setting.

Theorem 10.1. *Suppose that* $\mathbf{Y}$ *is a nontrivial* α*-stable random element of* $\mathbb{K}$*. Then,* $0 < \alpha < 1$ *and the Lévy measure* ν *of* $\mathbf{Y}$ *obeys the scaling condition*

$$\nu(aB) = a^{-\alpha}\nu(B), \quad a > 0,$$

for all Borel sets $B \subseteq \mathbb{K}$. Conversely, if ν is a σ-finite measure on $\mathbb{K}\backslash\{\mathcal{E}\}$ that obeys the scaling condition for $0 < \alpha < 1$ and satisfies (9.1)*, then ν is the Lévy measure of an α-stable random element.*

Proof. If $(\mathbf{X}_t)_{t\geqslant 0}$ is the Lévy process corresponding to $\mathbf{Y}$, then it is not difficult to check that the process $(a^{-\frac{1}{\alpha}}\mathbf{X}_{at})_{t\geqslant 0}$ has the same distribution as $(\mathbf{X}_t)_{t\geqslant 0}$, and the scaling condition for ν follows easily.

Write η for the push-forward of ν by the map diam. It is clear from the scaling condition and the property $\operatorname{diam}(a\mathcal{X}) = a\operatorname{diam}(\mathcal{X})$ that $\eta([x,\infty)) = cx^{-\alpha}$ for some constant $c > 0$, and so

$$\int (\operatorname{diam}(\mathcal{Y}) \wedge 1)\,\nu(d\mathcal{Y}) = \int (y \wedge 1)\, c\alpha y^{-(\alpha+1)}\, dy.$$

In order for this integral to be finite, it must be the case that $\alpha < 1$.

The remainder of the result is straightforward and we omit the proof. □

Remark 10.2*.* One of the conclusions of Theorem 10.1 is that there are no nontrivial α-stable random elements for $\alpha \geqslant 1$. This also follows from the following argument. If $\mathbf{Y}$ was a nontrivial α-stable random element and $(\mathbf{Y}_k)_{k\in\mathbb{N}}$ was a sequence of independent copies of $\mathbf{Y}$, then $\frac{1}{n^{\frac{1}{\alpha}}}\boxplus_{k=0}^{n-1}\mathbf{Y}_k$ would have the same distribution as $\mathbf{Y}$ and hence $\frac{1}{n}\boxplus_{k=0}^{n-1}\mathbf{Y}_k$ would certainly converge in distribution as $n \to \infty$, but this contradicts Remark 8.3, where we observed that there is no analogue of a law of large numbers in our setting.

By [DMZ08, Theorem 7.14], each α-stable random element $\mathbf{Y}$ can be represented as a *LePage series*

$$\mathbf{Y} \overset{d}{\sim} \gamma \boxplus_{n\in\mathbb{N}} \Gamma_n^{-\frac{1}{\alpha}}\mathbf{Z}_n, \tag{10.1}$$

where γ is a suitable constant, $(\Gamma_n)_{n\in\mathbb{N}}$ is the sequence of successive arrivals of a homogeneous unit intensity Poisson point process on $\mathbb{R}_+$ and $(\mathbf{Z}_n)_{n\in\mathbb{N}}$ is sequence of i.i.d. random elements of $\mathbb{K}$ with almost surely constant diameter 1.

For the sake of completeness, we give a quick self-contained proof of why this is so in our setting. From Theorem 9.1 and Theorem 10.1, $\mathbf{Y}$ has the same distribution as $\boxplus\{\mathcal{X} : (t,\mathcal{X}) \in \Pi\}$, where Π is a Poisson point process on $[0,1] \times (\mathbb{K}\backslash\{\mathcal{E}\})$ with intensity $\lambda \otimes \nu$ for a measure ν that has the property $\nu(aB) = a^{-\alpha}\nu(B)$, for all $a > 0$ and all Borel sets $B \subseteq \mathbb{K}$. From the proof of Theorem 10.1, we know that the values of $\operatorname{diam}(\mathcal{X})$ as we range over points $(t,\mathcal{X})$ in Π are distinct and form a Poisson point process with intensity measure η, where $\eta([x,\infty)) = cx^{-\alpha}$ for some constant $c > 0$. Similar reasoning shows that $\{(\operatorname{diam}(\mathcal{X}), \operatorname{diam}(\mathcal{X})^{-1}\mathcal{X}) : (t, \mathcal{X}) \in \Pi\}$ is a Poisson point process with intensity measure $\eta \otimes \kappa$ for some probability measure κ supported on $\{\mathcal{Y} \in \mathbb{K} : \operatorname{diam}(\mathcal{Y}) = 1\}$. It follows that $\{(c^{-\frac{1}{\alpha}} \operatorname{diam}(\mathcal{X})^{-\alpha}, \operatorname{diam}(\mathcal{X})^{-1}\mathcal{X}) : (t, \mathcal{X}) \in \Pi\}$ is a Poisson point process with intensity $\lambda \otimes \kappa$. If we denote this Poisson point process by Ξ and list the points of Ξ as $((\Gamma_n, \mathbf{Z}_n))_{n \in \mathbb{N}}$, where the first coordinates are in increasing order, then this sequence has the description given above and $\{\mathcal{X} : (t, \mathcal{X}) \in \Pi\} = \{c^{-\frac{1}{\alpha^2}} \Gamma_n^{-\frac{1}{\alpha}} \mathbf{Z}_n : n \in \mathbb{N}\}$, which establishes the result.

It is possible to reconstruct the Poisson process Π from the sequences $(\Gamma_n)_{n \in \mathbb{N}}$ and $(\mathbf{Z}_n)_{n \in \mathbb{N}}$, so that the uniqueness part of Theorem 9.1 yields that the common distribution of the random elements $\mathbf{Z}_n$ is unique. A sum of the form (10.1) produces an α-stable random element for any i.i.d. sequence $(\mathbf{Z}_n)_{n \in \mathbb{N}}$ with the property that the sum is well-defined which, by part (b) of Corollary 3.7, is equivalent to $\sum_n \Gamma_n^{-\frac{1}{\alpha}} \operatorname{diam}(\mathbf{Z}_n) < \infty$ almost surely (that is, it is not necessary to impose the condition that the diameters of the random elements $\mathbf{Z}_n$ are constant). If ρ is the common distribution of the random variables $\operatorname{diam}(\mathbf{Z}_n)$, then the latter sum converges almost surely if and only if $\int z^\alpha \, \rho(dz) < \infty$. If the diameters of the $\mathbf{Z}_n$ are not constant, then different common distributions for the $\mathbf{Z}_n$ may yield sums in (10.1) that have the same distribution.

Example 10.3. We consider the α-stable random element $\mathbf{Y}$ obtained by constructing the LePage series in which the $\mathbf{Z}_i$ are copies of some common nonrandom $\mathcal{Z} \in \mathbb{K}$. In this case $\mathbf{Y}$ is the infinite product $Y = Z^\infty$ equipped with the metric

$$r_Y((z_n'), (z_n'')) := \sum_n \Gamma_n^{-\frac{1}{\alpha}} r_Z(z_n', z_n'')$$

and the probability measure $\mu_Y := \mu_Z^{\otimes \infty}$. Note that $r_Y((z_n'), (z_n''))$ is a positive strictly stable random variable with index α.

11. Thinning

Recall the map Ψ that associates with each $\mathcal{X} \in \mathbb{K}$ a $\mathbb{N}$-valued measure on $\mathbb{I}$. For $p \in [0, 1]$, the independent p-thinning of an $\mathbb{N}$-valued measure $N := \sum_k m_k \delta_{\mathcal{Y}_k}$ is defined in the usual way as $N^{(p)} := \sum_k \xi_k \delta_{\mathcal{Y}_k}$, where ξ_k, $k \in \mathbb{N}$, are independent binomial random variables with parameters m_k and p. In other words, each atom of N is retained with probability p and otherwise eliminated independently of all other atoms and taking into account the multiplicities.

Applying an independent p-thinning procedure to the point process $\mathbf{N} := \Psi(\mathbf{X})$ generated by random element $\mathbf{X}$ in $\mathbb{K}$ yields a $\mathbb{K}$-valued random element $\mathbf{X}^{(p)} := \Sigma(\mathbf{N}^{(p)})$ that we call the *p-thinning* of $\mathbf{X}$. Note that the $\mathbf{X}^{(p)} \leqslant \mathbf{X}$, $\mathbf{X}^{(0)} = \mathcal{E}$, $\mathbf{X}^{(1)} = \mathbf{X}$, and it is possible to couple the constructions of these random elements so that $\mathbf{X}^{(p)} \leqslant \mathbf{X}^{(q)}$ for $0 \leqslant p \leqslant q \leqslant 1$.

For $0 \leqslant p, q \leqslant 1$ the random element $(\mathbf{X}^{(p)})^{(q)}$ has the same distribution as the random element $\mathbf{X}^{(pq)}$. Also, if $\mathbf{X}$ and $\mathbf{Y}$ are independent random elements and $\mathbf{X}^{(p)}$ and $\mathbf{Y}^{(p)}$ are constructed to be independent, then $\mathbf{X}^{(p)} \boxplus \mathbf{Y}^{(p)}$ has the same distribution as $(\mathbf{X} \boxplus \mathbf{Y})^{(p)}$. It follows from this last property that, for fixed $A \in \mathbb{A}$ and $0 \leqslant p \leqslant 1$, the map

$$\mathcal{X} \mapsto \mathbb{E}[\chi_A(\mathcal{X}^{(p)})] = \prod (1 - p + p\chi_A(\mathcal{Y}_n))$$

is a semicharacter, where the product ranges over the factors that appear in the factorization of $\mathcal{X}$ into a sum of irreducible elements of $\mathbb{K}$ (repeated, of course, according to their multiplicities). This is a particular case of the construction in Remark 5.10.

12. The Gromov-Prohorov metric

We follow the definition of the *Gromov-Prohorov* metric in [GPW09].

Recall that the distance in the *Prohorov metric* between two probability measures μ_1 and μ_2 on a common metric space (Z, r_Z) is defined by

$$d_{\mathrm{Pr}}^{(Z,r_Z)}(\mu_1, \mu_2) := \inf\{\varepsilon > 0 : \mu_1(F) \leqslant \mu_2(F^\varepsilon) + \varepsilon, \; \forall F \text{ closed}\},$$

where

$$F^\varepsilon := \{z \in Z : r_Z(z, z') < \varepsilon, \text{ for some } z' \in F\}.$$

An alternative characterization of the Prohorov metric due to Strassen (see, for example, [EK86, Theorem 3.1.2] or [Dud02, Corollary 11.6.4]) is that

$$d_{\mathrm{Pr}}^{(Z,r_Z)}(\mu_1, \mu_2) = \inf_\pi \inf\{\varepsilon > 0 : \pi\{(z, z') \in Z \times Z : r_Z(z, z') \geqslant \varepsilon\} \leqslant \varepsilon\},$$

where the infimum is over all probability measures π such that $\pi(\cdot \times Z) = \mu_1$ and $\pi(Z \times \cdot) = \mu_2$.

The following result is no doubt well-known, but we include it for completeness. Recall that if (X, r_X) and (Y, r_Y) are two metric spaces, then $r_X \oplus r_Y$ is the metric on the Cartesian product $X \times Y$ given by $r_X \oplus r_Y((x', y'), (x'', y'')) = r_X(x', x'') + r_Y(y', y'')$.

Lemma 12.1. *Suppose that μ_1 and μ_2 (resp. ν_1 and ν_2) are probability measures on a metric space (X, r_X) (resp. (Y, r_Y)). Then,*

$$d_{\mathrm{Pr}}^{(X\times Y, r_X\oplus r_Y)}(\mu_1\otimes\nu_1, \mu_2\otimes\nu_2) \leqslant d_{\mathrm{Pr}}^{(X,r_X)}(\mu_1,\mu_2) + d_{\mathrm{Pr}}^{(Y,r_Y)}(\nu_1,\nu_2).$$

Proof. This is immediate from the observation that if α and β are probability measures on $X\times X$ and $Y\times Y$, respectively, such that

$$\alpha\{(x',x'')\in X\times X : r_X(x',x'')\geqslant\gamma\}\leqslant\gamma$$

and

$$\beta\{(y',y'')\in Y\times Y : r_Y(y',y'')\geqslant\delta\}\leqslant\delta$$

for $\gamma,\delta>0$, then

$$\begin{aligned}\alpha\otimes\beta\{&((x',y'),(x'',y''))\in(X\times Y)\times(X\times Y)\\ &: r_X(x',x'')+r_Y(y',y'')\geqslant\gamma+\delta\}\\ &\leqslant\gamma+\delta,\end{aligned}$$

where, with a slight abuse of notation, we identify the measure $\alpha\otimes\beta$ on $(X\times X)\times(Y\times Y)$ with its push-forward on $(X\times Y)\times(X\times Y)$ by the map $((x',x''),(y',y''))\mapsto((x',y'),(x'',y''))$. ☐

The following lemma is also probably well-known.

Lemma 12.2. *Suppose that μ_1 and μ_2 are two probability measures on a compact metric space (X, r_X) and ν is a probability measure on another compact metric space (Y, r_Y). Then,*

$$d_{\mathrm{Pr}}^{(X\times Y, r_X\oplus r_Y)}(\mu_1\otimes\nu, \mu_2\otimes\nu) = d_{\mathrm{Pr}}^{(X,r_X)}(\mu_1,\mu_2).$$

Proof. It follows from Lemma 12.1 that

$$\begin{aligned}d_{\mathrm{Pr}}^{(X\times Y, r_X\oplus r_Y)}(\mu_1\otimes\nu, \mu_2\otimes\nu) &\leqslant d_{\mathrm{Pr}}^{(X,r_X)}(\mu_1,\mu_2) + d_{\mathrm{Pr}}^{(Y,r_Y)}(\nu,\nu)\\ &= d_{\mathrm{Pr}}^{(X,r_X)}(\mu_1,\mu_2).\end{aligned}$$

On the other hand, suppose that π is a probability measure on $(X\times Y)\times(X\times Y)$ such that $\pi(\cdot\times(X\times Y))=\mu_1\otimes\nu$, $\pi((X\times Y)\times\cdot)=\mu_2\otimes\nu$ and

$$\pi\{((x',y'),(x'',y''))\in(X\times Y)\times(X\times Y) : r_X(x',x'')+r_Y(y',y'')\geqslant\varepsilon\}\leqslant\varepsilon$$

for some $\varepsilon>0$. If ρ is the push-forward of π by the map $((x',y'),(x'',y''))\mapsto(x',x'')$, then it is clear that $\rho(\cdot\times X)=\mu_1$, $\rho(X\times\cdot)=\mu_2$ and

$$\rho\{((x',x'')\in X\times X : r_X(x',x'')\geqslant\varepsilon\}\leqslant\varepsilon,$$

and hence

$$= d_{\mathrm{Pr}}^{(X,r_X)}(\mu_1,\mu_2)\leqslant d_{\mathrm{Pr}}^{(X\times Y, r_X\oplus r_Y)}(\mu_1\otimes\nu, \mu_2\otimes\nu).$$

□

The *Gromov-Prohorov metric* is a metric on the space of equivalence classes of compact metric measure space (recall that two compact metric measure spaces are equivalent if there is an isometry mapping one to the other such that the probability measure on the first is mapped to the probability measure on the second). Given two compact metric measure spaces $\mathcal{X} = (X, r_X, \mu_X)$ and $\mathcal{Y} = (Y, r_Y, \mu_Y)$, the Gromov-Prohorov distance between their equivalence classes is

$$d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{Y}) := \inf_{(\phi_X, \phi_Y, Z)} d_{\mathrm{Pr}}^{(Z, r_Z)}((\phi_X)_\# \mu_X, (\phi_Y)_\# \mu_Y),$$

where the infimum is taken over all compact metric spaces (Z, r_Z) and isometric embeddings ϕ_X of X and ϕ_Y of Y into Z, and $(\phi_X)_\# \mu_X$ (resp. $(\phi_Y)_\# \mu_Y$) denotes the push-forward of μX by ϕ_X (resp. μ_Y by ϕ_Y).

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