The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *

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Abstract

The two-parameter Poisson-Dirichlet distribution, denoted $\mathrm{PD}(\alpha, \theta)$, is a distribution on the set of decreasing positive sequences with sum 1. The usual Poisson-Dirichlet distribution with a single parameter θ, introduced by Kingman, is $\mathrm{PD}(0, \theta)$. Known properties of $\mathrm{PD}(0, \theta)$, including the Markov chain description due to Vershik-Shmidt-Ignatov, are generalized to the two-parameter case. The size-biased random permutation of $\mathrm{PD}(\alpha, \theta)$ is a simple residual allocation model proposed by Engen in the context of species diversity, and rediscovered by Perman and the authors in the study of excursions of Brownian motion and Bessel processes. For $0 < \alpha < 1$, $\mathrm{PD}(\alpha, 0)$ is the asymptotic distribution of ranked lengths of excursions of a Markov chain away from a state whose recurrence time distribution is in the domain of attraction of a stable law of index α. Formulae in this case trace back to work of Darling, Lamperti and Wendel in the 1950's and 60's. The distribution of ranked lengths of excursions of a one-dimensional Brownian motion is $\mathrm{PD}(1/2, 0)$, and the corresponding distribution for Brownian bridge is $\mathrm{PD}(1/2, 1/2)$. The $\mathrm{PD}(\alpha, 0)$ and $\mathrm{PD}(\alpha, \alpha)$ distributions admit a similar interpretation in terms of the ranked lengths of excursions of a semi-stable Markov process whose zero set is the range of a stable subordinator of index α.

Contents

1 Introduction

The subject of this paper is a two-parameter family of probability distributions for a sequence of random variables

$$(V_n) = (V_1, V_2, \cdots) \text{ with } V_1 > V_2 > \cdots > 0 \text{ and } \sum_n V_n = 1 \text{ a.s.} \qquad (1)$$

This family is an extension of the one-parameter family of Poisson-Dirichlet distributions, introduced by Kingman [33] and denoted here by $\{\text{PD}(0, \theta), \theta >$

0}, which arises from the study of asymptotic distributions of random ranked relative frequencies in a variety of contexts including number theory [57], combinatorics [58, 3, 25], and population genetics [63, 19]. Study of an enlarged family, involving another parameter α with $0 \leq \alpha < 1$, is motivated by parallels between $\text{PD}(0, \theta)$ and the asymptotic distributions of ranked relative lengths of intervals derived in renewal theory from lifetime distributions in the domain of attraction of a stable law of index α [37, 65]. As explained in Section 1.2, this family of asymptotic distributions for (V_n) as in (1), denoted here by $\{\text{PD}(\alpha, 0), 0 < \alpha < 1\}$, can be interpreted in terms of ranked lengths of excursion intervals between zeros of B, where B is Brownian motion for $\alpha = \frac{1}{2}$, or a recurrent Bessel process of dimension $2 - 2\alpha$ for $0 < \alpha < 1$. By a change of measure relative to $\text{PD}(\alpha, 0)$, with a density depending on θ described in Proposition 13, we can define $\text{PD}(\alpha, \theta)$ for arbitrary $0 < \alpha < 1$ and $\theta > -\alpha$, then recover Kingman's Poisson-Dirichlet distribution $\text{PD}(0, \theta)$ for $\theta > 0$ as the weak limit of $\text{PD}(\alpha, \theta)$ as $\alpha \downarrow 0$. We prefer however to present a unified definition of $\text{PD}(\alpha, \theta)$ as follows.

1.1 The size-biased permutation of $\text{PD}(\alpha, \theta)$

The following definition originates from the application of random discrete distributions to model the division of a large population into a large number of possible species or types. A ranked sequence of random frequencies (V_n) as in (1) represents the structure of an idealized infinite population that has been randomly partitioned into various species. Then V_n represents the proportion of the population that belongs to the nth most common species. See [17, 33, 19, 16, 44, 46] for background and further references to such applications. The *size-biased permutation* of (V_n) is the sequence of proportions of species in their order of appearance in a process of random sampling from the population. This notion is made precise as follows. For (V_n) as in (1), call a random variable $\tilde{V}_1$ a *size-biased pick from* (V_n) if

$$P(\tilde{V}_1 = V_n | V_1, V_2, \cdots) = V_n, \quad (n = 1, 2, \cdots) \tag{2}$$

Here $\tilde{V}_1$ may be already defined on the same probability space as (V_n), or constructed by additional randomization on an enlarged probability space. Similarly, call $(\tilde{V}_1, \tilde{V}_2, \cdots)$ a *size-biased permutation* of (V_n) if $\tilde{V}_1$ is a size-biased pick from (V_n), and for each $n = 1, 2, \cdots, j = 1, 2, \cdots,$

$$P(\tilde{V}_{n+1} = V_j | \tilde{V}_1, \cdots, \tilde{V}_n; V_1, V_2, \cdots) = \frac{V_j 1(V_j \neq \tilde{V}_i \text{ for some } 1 \leq i \leq n)}{(1 - \tilde{V}_1 - \cdots - \tilde{V}_n)}$$

Following Engen [17], Perman-Pitman-Yor [44], we make the following definition in terms of independent beta random variables. See also Sections 9.1 and 9.2 for further motivation. Recall that for $a > 0$, $b > 0$, the *beta(a, b) distribution* on $(0, 1)$ has density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \qquad (0 < x < 1) \tag{3}$$

Definition 1 For $0 \leq \alpha < 1$ and $\theta > -\alpha$, suppose that a probability $P_{\alpha,\theta}$ governs independent random variables $\tilde{Y}_n$ such that $\tilde{Y}_n$ has beta$(1-\alpha, \theta+n\alpha)$ distribution. Let

$$\tilde{V}_1 = \tilde{Y}_1, \quad \tilde{V}_n = (1 - \tilde{Y}_1) \cdots (1 - \tilde{Y}_{n-1}) \tilde{Y}_n \qquad (n \geq 2) \tag{4}$$

and let $V_1 \geq V_2 \geq \cdots$ be the ranked values of the $\tilde{V}_n$. Define the *Poisson-Dirichlet distribution with parameters (α, θ)*, or $\text{PD}(\alpha, \theta)$ to be the $P_{\alpha,\theta}$ distribution of (V_n).

Results of [44] show that this definition of $\text{PD}(\alpha, \theta)$ agrees with the previous descriptions of $\text{PD}(0, \theta)$ and $\text{PD}(\alpha, 0)$, and yield the following result:

Proposition 2 [44, 46] *Under $P_{\alpha,\theta}$ governing $(\tilde{Y}_n)$, $(\tilde{V}_n)$ and (V_n) as in Definition 1, $(\tilde{V}_n)$ is a size-biased permutation of (V_n).*

For $\alpha = 0$ this result was obtained by McCloskey [41]. Ewens [19] calls the distribution of $(\tilde{V}_n)$ defined by (4) the GEM distribution, after Griffiths, Engen and McCloskey. Engen [17] considered the residual allocation model (4) for $(\tilde{V}_n)$ for $0 \leq \alpha < 1$ and $\theta > 0$, and established the consequence of Proposition 2 that for (V_n) with $\text{PD}(\alpha, \theta)$ distribution, a size-biased pick $\tilde{V}_1$ from (V_n) has beta$(1 - \alpha, \theta + \alpha)$ distribution. As noted by Engen, this gives the following formula for all non-negative measurable functions f:

$$E_{\alpha,\theta} \sum_{n=1}^{\infty} f(V_n) = E_{\alpha,\theta} \tilde{V}_1^{-1} f(\tilde{V}_1) = \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)} \int_0^1 du f(u) \frac{(1-u)^{\alpha+\theta-1}}{u^{\alpha+1}} \tag{5}$$

where $E_{\alpha,\theta}$ denotes expectation with respect to the probability distribution $P_{\alpha,\theta}$ in Definition 1.

The particular choice of beta distributions for $\tilde{Y}_n$ in Definition 1, and the consequent parameter set $\{0 \leq \alpha < 1, \theta > -\alpha\}$ for the two-parameter Poisson-Dirichlet distribution, is dictated by the following result, which generalizes a well known characterization of $\text{PD}(0,\theta)$ due to McCloskey [41]. See [46] for variations and further references.

Proposition 3 [46] *For (V_n) as in (1), a size-biased random permutation $(\tilde{V}_n)$ of (V_n) admits the expression (4) for a sequence of independent random variables $(\tilde{Y}_n)$ iff the distribution of the $\tilde{Y}_n$ is of the form assumed in Definition 1, that is iff (V_n) has $\text{PD}(\alpha,\theta)$ distribution for some $0 \leq \alpha < 1$ and $\theta > -\alpha$.*

1.2 Interval lengths derived from a subordinator

Following Lamperti [37], Wendel [65], Kingman [33], Perman-Pitman-Yor [43, 44, 50], consider the sequence

$$V_1(T) \geq V_2(T) \geq \cdots \tag{6}$$

of ranked lengths of component intervals of the set $[0,T]\backslash Z$, where Z is a random closed subset of $[0,\infty)$ with Lebesgue measure 0, and T is a strictly positive random time. Suppose Z is the closure of the range of a *subordinator* $(\tau_s, s \geq 0)$, i.e. an increasing process with stationary independent increments. We assume that (τ_s) has no drift component, so

$$E[\exp(-\lambda\tau_s)] = \exp\left(-s\int_0^\infty (1-e^{-\lambda x})\Lambda(dx)\right) \tag{7}$$

where the Lévy measure Λ on $(0,\infty)$ is the intensity measure for the Poisson point process of jumps $(\tau_s - \tau_{s-}, s \geq 0)$. Call (τ_s) a *gamma subordinator* if $\Lambda(dx) = x^{-1}e^{-x}dx, x > 0$, that is if τ_s has the *gamma(s) distribution*

$$P(\tau_s \in dx) = \Gamma(s)^{-1}x^{s-1}e^{-x}\,dx \qquad (x > 0), \tag{8}$$

for each $s > 0$. There is the following well known representation of $\text{PD}(0,\theta)$:

Proposition 4 [33] *If* (τ_s) *is a gamma subordinator then for every* $\theta > 0$

$$\left(\frac{V_1(\tau_\theta)}{\tau_\theta}, \frac{V_2(\tau_\theta)}{\tau_\theta}, \cdots\right) \text{ has PD}(0,\theta) \text{ distribution} \tag{9}$$

independently of τ_θ*.*

Call (τ_s) a *stable subordinator of index* α, where $0 < \alpha < 1$, if $\Lambda = \Lambda_\alpha$ where

$$\Lambda_\alpha(x,\infty) = Cx^{-\alpha} \qquad (x > 0) \tag{10}$$

for some constant $C > 0$. That is, from (7), for $\lambda > 0$

$$E[\exp(-\lambda\tau_s)] = \exp(-sK\lambda^\alpha) \text{ where } K = C, (1-\alpha). \tag{11}$$

The following companion of Proposition 4 plays a key role in this paper. The equality in distribution of the two sequences displayed in (12) and (13) was established in [50], while the connection with Definition 1 was made in [44]. See also [5, 50, 62, 51] regarding the relation between the this description of PD$(\alpha, 0)$ and the generalized arc-sine laws of Lamperti [36].

Proposition 5 [44, 50] *If* (τ_s) *is a stable* (α) *subordinator for some* $0 < \alpha < 1$ *then for every* $s > 0$

$$\left(\frac{V_1(\tau_s)}{\tau_s}, \frac{V_2(\tau_s)}{\tau_s}, \cdots\right) \text{ has PD}(\alpha,0) \text{ distribution} \tag{12}$$

and also for every fixed $t > 0$

$$\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}, \cdots\right) \text{ has PD}(\alpha,0) \text{ distribution} \tag{13}$$

Since the zero set Z of a standard one-dimensional Brownian motion B is the closure of the range of a stable $(\frac{1}{2})$ subordinator [40], (13) shows that PD$(\frac{1}{2}, 0)$ is the distribution of the ranked lengths of the excursions of B away from 0 during the time interval $[0, 1]$, where the lengths include the length $1 - G_1$ of the final *meander interval*, where

$$G_t = \sup([0,t) \cap Z) = \sup\{s : s < t, B_s = 0\} \tag{14}$$

$\mathrm{PD}(\alpha, 0)$ can be interpreted similarly, in terms of the ranked lengths of excursion intervals, if the Brownian motion B is replaced by a suitable semi-stable Markov process [38], for example a Bessel process of dimension $\delta = 2 - 2\alpha$ [42], or, for $0 < \alpha < \frac{1}{2}$, a stable Lévy process of index $1/(1-\alpha)$ [21].

The $\mathrm{PD}(\alpha, \alpha)$ distribution arises naturally as the distribution of ranked lengths of excursions of a semi-stable Markov bridge derived from a Markov process whose zero set is the range of a stable (α) subordinator [65, 50, 44]. It is well known that such a bridge can be derived from the unconditioned process on interval $[0, G_t]$ by appropriate scaling. So as a companion to (12) and (13), in the same setting we have for each fixed $t > 0$,

$$\left(\frac{V_1(G_t)}{G_t}, \frac{V_2(G_t)}{G_t}, \cdots\right) \textit{ has } \mathrm{PD}(\alpha, \alpha) \textit{ distribution} \tag{15}$$

independently of G_t. In particular, we note the following:

Proposition 6 [44, 50] *If V_n is the length of the nth longest excursion of B away from 0 over the time interval $[0, 1]$, then*

$$(V_n) \textit{ has } \mathrm{PD}(\tfrac{1}{2}, 0) \textit{ distribution if } B \textit{ is Brownian motion;} \tag{16}$$

$$(V_n) \textit{ has } \mathrm{PD}(\tfrac{1}{2}, \tfrac{1}{2}) \textit{ distribution if } B \textit{ is Brownian bridge.} \tag{17}$$

Stepanov [55] encountered asymptotics involving $\mathrm{PD}(\frac{1}{2}, \frac{1}{2})$ in the study of the asymptotic distribution of the sizes of tree components in a random mapping. The connection with Brownian bridge in this setting is explained in Aldous-Pitman [1].

The $\mathrm{PD}(\alpha, 0)$ distribution also arises as the asymptotic distribution of

$$\left(\frac{V_1(T)}{T}, \frac{V_2(T)}{T}, \cdots\right) \tag{18}$$

either for fixed T as $T \to \infty$, or for $T = \tau_s$ as $s \to \infty$, for any subordinator (τ_s) such that $\Lambda(x, \infty) = x^{-\alpha} L(x)$ as $x \to \infty$ for a slowly varying function $L(x)$. Similarly, $\mathrm{PD}(\alpha, 0)$ is the asymptotic distribution as $n \to \infty$ of

$$\left(\frac{X_{(n,1)}}{S_n}, \frac{X_{(n,2)}}{S_n}, \cdots, \frac{X_{(n,n)}}{S_n}\right) \tag{19}$$

for $X_{(n,1)} \geq X_{(n,2)} \geq \cdots \geq X_{(n,n)}$ the order statistics of of i.i.d positive random variables $X_1, \cdots, X_n$ with sum S_n, assuming $P(X_i \geq x) = x^{-\alpha} L(x)$ as $x \to \infty$. Related results have been studied by many authors: see for instance [12, 4, 37, 29, 30, 52]. Many limit distributions found in these papers are exact distributions of various functions of a PD$(\alpha, 0)$ sequence. For instance, Darling [12] found the characteristic function of the limiting distribution of $S_n / X_{(n,1)}$ in (19). This is the characteristic function of $1/V_1$ for a PD$(\alpha, 0)$ sequence (V_n). Lamperti [37] derived the corresponding Laplace transform, given by (37) of this paper with $n = 1$, from the asymptotic distribution as $n \to \infty$ of the maximum up to time n of the age process derived from a discrete renewal process with lifetime distribution in the domain of attraction of a stable law of index α. That the same transform appears in both Darling's and Lamperti's works amounts to the equality in distribution of first components in (12) and (13). The equality in distribution of the first n components in (12) and (13) can be interpreted similarly as an asymptotic result in renewal theory.

1.3 Organization

The main theme of this paper is the development of various results for PD(α, θ) in the general two parameter case. Mostly these results are known in one or other of the special cases $\alpha = 0$ or $\theta = 0$. Many results acquire their simplest form for PD$(\alpha, 0)$ with $0 < \alpha < 1$. These results for PD$(\alpha, 0)$ are presented in Section 2, followed by results for PD(α, θ) in Section 3. These two sections also serve as a guide to the rest of the paper, which contains proofs of the results in Sections 2 and 3, and various further developments.

2 Main results for PD$(\alpha, 0)$

Results stated in this Section are proved in Section 4.

Proposition 7 *Suppose* (V_n) *has* PD$(\alpha, 0)$ *distribution for some* $0 < \alpha < 1$. *Let*

$$R_n = \frac{V_{n+1}}{V_n} \tag{20}$$

Then R_n *has beta*$(n\alpha, 1)$ *distribution, that is*

$$P(R_n \leq r) = r^{n\alpha}, \qquad (0 \leq r \leq 1) \tag{21}$$

and the R_n are mutually independent.

Since (V_n) can be recovered from (R_n) as

$$V_1 = \frac{1}{1 + R_1 + R_1R_2 + R_1R_2R_3 + \cdots}; \qquad V_{n+1} = V_1R_1R_2\cdots R_n \quad (n \geq 1), \tag{22}$$

the following simple construction of $\mathrm{PD}(\alpha, 0)$ is an immediate corollary of Proposition 7.

Corollary 8 *Suppose (R_n) is a sequence of independent random variables such that R_n has beta$(n\alpha, 1)$ distribution, for some $0 < \alpha < 1$. Then (V_n) defined by (22) has $\mathrm{PD}(\alpha, 0)$ distribution.*

The next proposition summarizes and sharpens some results from [33, 50]:

Proposition 9 *Suppose (V_n) has $\mathrm{PD}(\alpha, 0)$ distribution for some $0 < \alpha < 1$.*

(i) *The limit*

$$L := \lim_{n\to\infty} nV_n^\alpha \tag{23}$$

exists both almost surely and in pth mean for all $p > 0$.

(ii) *Let*

$$\Sigma := (L/C)^{-1/\alpha}, \qquad \Delta_n := V_n\Sigma. \tag{24}$$

Then Σ has the same stable (α) distribution as τ_1 in (11), the Δ_n are the ranked points of a PRM Λ_α on $(0, \infty)$, where $\Lambda_\alpha(x, \infty) = Cx^{-\alpha}$ for $x > 0$, and (V_n) may be represented as

$$V_n = \Delta_n/\Sigma \text{ where } \Sigma = \sum_n \Delta_n \tag{25}$$

(iii) *Let*

$$X_n := \Lambda_\alpha(\Delta_n, \infty) = C\Delta_n^{-\alpha} = LV_n^{-\alpha} \tag{26}$$

Then the $X_1 < X_2 < \cdots$ are the points of a PRM (dx) on $(0, \infty)$, i.e.

$$X_n = \epsilon_1 + \cdots + \epsilon_n \tag{27}$$

where the ϵ_i are independent standard exponential variables, and (V_n) may be represented in terms of (X_n) as

$$V_n = \frac{X_n^{-1/\alpha}}{\sum_m X_m^{-1/\alpha}} \tag{28}$$

In the setting described of Section 1.2, where $V_n = V_n(1)$ is the nth longest subinterval in the complement of $[0,1]\backslash Z$ where Z is the zero set of a semi-stable Markov process X, L is a multiple of the local time of X at zero up to time 1. See [51] for the corresponding result when $V_n = V_n(T)/T$ for suitable random T. The distribution of $L = C\Sigma^{-\alpha}$ is determined by its moments

$$E(L^p) = C^p E(\Sigma^{-\alpha p}) = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)}\Gamma(1-\alpha)^{-p} \qquad (p > -1) \tag{29}$$

So $\Gamma(1-\alpha)L$ has the Mittag-Leffler (α) distribution [21, 42, 7, 44]. The joint distribution of L and $V_1, \cdots, V_n$ can be read from that of Σ and $V_1, \cdots, V_n$, which is described in Proposition 45. In formula (28), which serves to construct a PD$(\alpha, 0)$ sequence (V_n) from a sequence of independent standard exponential variables (ϵ_n), the denominator has a stable(α) distribution. This method of constructing a random variable with infinitely divisible distribution from the ranked jumps of its Poisson representation, originally due to Lévy, has been exploited in several contexts [61, 39].

The next Proposition exposes results underlying a formula for the Laplace transform of $1/V_n$ that is stated in Corollary 11 following the proposition. This formula was obtained in different settings by Darling [12] and Lamperti [37] for $n = 1$ and Wendel [65] for $n = 2, 3, \cdots$. See also Horowitz [30], Kingman [33], Resnick [52].

Proposition 10 *Suppose* (V_n) *has* PD$(\alpha, 0)$ *distribution for some* $0 < \alpha < 1$. *Let* $A_0 = 0$ *and for* $n = 1, 2, \cdots$ *define random variables* A_n *and* Σ_n *by*

$$A_n := \frac{V_1 + V_2 + \cdots + V_n}{V_{n+1}} = \frac{1}{R_n} + \frac{1}{R_n R_{n-1}} + \cdots + \frac{1}{R_n R_{n-1} \cdots R_1}, \tag{30}$$

$$\Sigma_n := \frac{V_{n+1} + V_{n+2} + \cdots}{V_n} = \frac{1 - V_1 - \cdots - V_n}{V_n} = R_n + R_n R_{n+1} + R_n R_{n+1} R_{n+2} + \cdots. \tag{31}$$

where $R_n = V_{n+1}/V_n$ *as in Proposition 7. For* $\lambda \geq 0$ *let*

$$\phi_\alpha(\lambda) := \alpha \int_1^\infty dx\, e^{-\lambda x} x^{-\alpha-1} \tag{32}$$

$$\psi_\alpha(\lambda) := 1 + \alpha \int_0^1 dx\, (1 - e^{-\lambda x}) x^{-\alpha-1} = \Gamma(1-\alpha)\lambda^\alpha + \phi_\alpha(\lambda) \tag{33}$$

Then

$$\frac{1}{V_n} = 1 + A_{n-1} + \Sigma_n \qquad \textit{where} \tag{34}$$

(i) A_{n-1} *is distributed as the sum of* $n-1$ *independent copies of* A_1*, with*

$$E[\,\exp(-\lambda A_{n-1})\,] = \phi_\alpha(\lambda)^{n-1} \tag{35}$$

(ii) Σ_n *is distributed as the sum of* n *independent copies of* Σ_1 *with*

$$E[\exp(-\lambda \Sigma_n)] = \psi_\alpha(\lambda)^{-n} \tag{36}$$

(iii) A_{n-1} *and* Σ_n *are independent.*

Corollary 11 [12, 37, 65] *If* (V_n) *has* PD$(\alpha, 0)$ *distribution, then the distribution of* V_n *is determined by the Laplace transform*

$$E[\exp(-\lambda/V_n)] = e^{-\lambda}\phi_\alpha(\lambda)^{n-1}\psi_\alpha(\lambda)^{-n} \tag{37}$$

For $V_n = V_n(1)$ derived from the interval lengths $V_n(t)$ generated by the range of a stable (α) subordinator, Wendel obtained (37) by considering the random times

$$H_n := \inf\{t : V_n(t) = 1\} \tag{38}$$

for $n = 1, 2, \cdots$, and using the identity in distribution

$$H_n \stackrel{d}{=} \frac{1}{V_n} \tag{39}$$

which follows by scaling from the equality of events

$$(H_n > t) = (V_n(t) < 1)$$

While both (H_n^{-1}) and (V_n) are decreasing random sequences, and (H_n^{-1}) has the same one-dimensional distributions as (V_n), this identity does not extend even to two-dimensional distributions, due to the fact that $\sum_n V_n = 1$ while there is no such constraint on $\sum_n H_n^{-1}$. However, comparison of Wendel's argument with our derivation of Proposition 10 reveals a remarkable extension of the identity in distribution (39):

Proposition 12 *For each* $n = 1, 2, \cdots$

$$\left(\frac{V_1(H_n)}{H_n}, \frac{V_2(H_n)}{H_n}, \cdots\right) \text{ has PD}(\alpha, 0) \text{ distribution.} \tag{40}$$

See also [51] for a generalization of Propositions 5 and 12.

Several authors have studied questions related to the a.s. limiting behaviour of $V_n(t)$ as $t \to \infty$ for $V_n(t)$ derived from the range of a stable subordinator. See e.g. Chung-Erdös [10], Csaki-Erdös-Revesz [11]. See Hu-Shi [31] for a number of refinements obtained using results of this paper.

3 Main results for PD(α, θ)

Results stated in this Section are proved in Section 5 except where otherwise indicated. For $0 \le \alpha < 1$ and $\theta > -\alpha$ let $E_{\alpha,\theta}$ denote expectation with respect to the probability $P_{\alpha,\theta}$ governing $(\tilde{V}_n)$ and (V_n) as in Definition 1. So the $P_{\alpha,\theta}$ distribution of (V_n) is PD(α, θ) .

3.1 Change of measure formulae

The basis for most of our computations for PD(α, θ) with $0 < \alpha < 1$ is the following Proposition, according to which the PD(α, θ) distribution admits a density relative to the PD$(\alpha, 0)$ distribution that is just a constant times $L^{\theta/\alpha}$ where L is the local time variable introduced in Proposition 9.

Proposition 13 [44] *Let* $0 < \alpha < 1$ *and* $\theta > -\alpha$. *For every non-negative product measurable function* f,

$$E_{\alpha,\theta}[\, f(V_1, V_2, \cdots)] = C_{\alpha,\theta} E_{\alpha,0}[\, L^{\theta/\alpha} f(V_1, V_2, \cdots)] \tag{41}$$

where $L := \lim_{n\to\infty} n V_n^{\alpha}$ *as in* (23), *and*

$$C_{\alpha,\theta} = \frac{1}{E_{\alpha,0}(L^{\theta/\alpha})} = \frac{\Gamma(\theta+1)}{\Gamma(\frac{\theta}{\alpha}+1)} \Gamma(1-\alpha)^{\theta/\alpha} \tag{42}$$

This is a re-expression of Corollary 3.15 of [44] using definitions made in this paper. The constant $C_{\alpha,\theta}$ is determined by (29). See also [47, 50, 45] for various alternative expressions for L.

Proposition 13 can be reformulated in various ways using different descriptions of PD$(\alpha, 0)$. For example, in the setting of Proposition 5, with $V_n(\tau_1)$ the nth largest jump of a stable (α) subordinator (τ_s) over $0 \leq s \leq 1$, we obtain

$$E_{\alpha,\theta}[\, f(V_1, V_2, \cdots)] = c_{\alpha,\theta} E\left[\tau_1^{-\theta} f\left(\frac{V_1(\tau_1)}{\tau_1}, \frac{V_2(\tau_1)}{\tau_1}, \cdots\right)\right] \tag{43}$$

where $c_{\alpha,\theta} = C^{\theta/\alpha} C_{\alpha,\theta}$ for $C_{\alpha,\theta}$ as in (42).

Proposition 13 shows that for fixed α with $0 < \alpha < 1$ the PD(α, θ) distributions are mutually absolutely continuous as θ varies. By contrast, for $\alpha = 0$ it is well known that the PD$(0, \theta)$ distributions are mutually singular as θ varies. Due to the way the definition of the local time variable L depends on α, the PD$(\alpha, 0)$ distributions are mutually singular as α varies, hence so too are the PD(α, θ) distributions for any fixed θ.

In Section 7 we obtain the following result, which generalizes both the Markov chain description of PD$(0, \theta)$ due to Vershik and Shmidt [58, 59] and Ignatov [32], and Proposition 7 for PD$(\alpha, 0)$.

Theorem 14 *Let*

$$Y_n = V_n/(V_n + V_{n+1} + \cdots) \qquad \textit{so} \tag{44}$$

$$V_1 = Y_1, \quad V_n = (1 - Y_1) \cdots (1 - Y_{n-1}) Y_n \qquad (n \geq 2) \tag{45}$$

Let $R_n = V_{n+1}/V_n$. For $0 \leq \alpha < 1, \theta > -\alpha$, let $P_{\alpha,\theta}$ govern (V_n) according to the PD(α, θ) *distribution, and let $P^*_{\alpha,\theta}$ govern $(R_1, R_2, \cdots)$ as a sequence of independent random variables, such that R_n has beta$(\theta + n\alpha, 1)$ distribution. Then*

$$E_{\alpha,\theta}[\, f(Y_1, Y_2, \cdots)\,] = K_{\alpha,\theta} E^*_{\alpha,\theta}[Y_1^\theta f(Y_1, Y_2, \cdots)] \tag{46}$$

for a constant $K_{\alpha,\theta}$. Both $P_{\alpha,\theta}$ and $P^*_{\alpha,\theta}$ govern (Y_n) as a Markov chain with the same forwards transition probabilities.

The chain (Y_n) is stationary and homogeneous under $P^*_{0,\theta}$, but for $0 < \alpha < 1$ the chain is non-homogeneous, and the distribution of Y_n depends on n, in a manner described precisely in Section 7.

According to Proposition 7, under $P_{\alpha,0}$ for $0 < \alpha < 1$ the ratios $R_n := V_{n+1}/V_n$ are mutually independent. Under $P_{\alpha,\theta}$ for $\theta \neq 0$ this is no longer true. However, it follows from Proposition 14 that under $P_{\alpha,\theta}$ the R_n are asymptotically independent for large n with beta$(\theta + n\alpha, 1)$ distributions. There is also the following formula for the joint density of $R_1, \cdots, R_n$:

Proposition 15 *Suppose* $0 < \alpha < 1$, $\theta > -\alpha$, *and* $\theta \neq 0$. *For* $0 < r_i < 1, i = 1, 2, \ldots, n$,

$$\frac{P_{\alpha,\theta}(R_1 \in dr_1, \ldots, R_n \in dr_n)}{dr_1 \cdots dr_n} = C_{\alpha,\theta}\alpha^n \Phi_\alpha(n + \frac{\theta}{\alpha}, \theta, a_n)\Pi_{i=1}^n r_i^{i\alpha-1}$$

where

$$a_n = \frac{1}{r_n} + \frac{1}{r_n\, r_{n-1}} + \cdots + \frac{1}{r_n \cdots r_1}$$

and the function Φ_α *is defined by*

$$\Phi_\alpha(\ell, \gamma, a) := \frac{\Gamma(\ell+1)}{\Gamma(\gamma)} \int_0^\infty dt\, t^{\gamma-1} e^{-t-at} \psi_\alpha(t)^{-\ell-1} = E[\, L^\ell\, V_1^{\gamma-\alpha\ell}\, (1 + aV_1)^{-\gamma}\,] \tag{47}$$

3.2 One-dimensional distributions

As an application of Proposition 13 we obtain the following formula for moments of the one-dimensional marginals of a PD(α, θ) distributed sequence:

Proposition 16 *For* $0 < \alpha < 1, \theta > -\alpha, p > 0, n = 1, 2 \cdots$

$$E_{\alpha,\theta}(V_n^p) = \frac{\Gamma(1-\alpha)^{\frac{\theta}{\alpha}}}{\Gamma(n)} \frac{\Gamma(\theta+1)}{\Gamma(\theta+p)} \frac{\Gamma(\frac{\theta}{\alpha}+n)}{\Gamma(\frac{\theta}{\alpha}+1)} \int_0^\infty dt\, t^{p+\theta-1} e^{-t} \phi_\alpha(t)^{n-1} \psi_\alpha(t)^{-\frac{\theta}{\alpha}-n} \tag{48}$$

where $\psi_\alpha(t)$ *and* $\phi_\alpha(t)$ *are as in* (36) *and* (35).

The following asymptotics as $n \to \infty$ are consequences of (23): for $0 < \alpha < 1$, $\theta > -\alpha$, $p > 0$,

$$n^{p/\alpha}\, E_{\alpha,\theta}(V_n^p) \to \frac{C_{\alpha,\theta}}{C_{\alpha,\theta+p}} \tag{49}$$

where $C_{\alpha,\theta}$ is defined by (42), and the right side of (49) is the pth moment of the $P_{\alpha,\theta}$ almost sure limit of $n^{1/\alpha}V_n$, that is $L^{1/\alpha}$. Note from (41) that the $P_{\alpha,\theta}$ distribution of L has a strictly positive density $f_{\alpha,\theta}$ on $(0, \infty)$ given by $f_{\alpha,\theta}(\ell) = C_{\alpha,\theta}\ell^{\theta/\alpha} f_{\alpha,0}(\ell)$ where $f_{\alpha,0}(\ell)$ is determined by the Mittag-Leffler density of the $P_{\alpha,0}$ distribution of $\Gamma(1-\alpha)L$, as discussed below (29).

By passage to the limit as $\alpha \to 0$ (see Section 5.2), we recover the known formula for $\mathrm{PD}(0,\theta)$:

Corollary 17 [54, 64, 23, 43]

$$E_{0,\theta}(V_n^p) = \frac{\Gamma(\theta)}{\Gamma(\theta+p)} \frac{\theta^n}{\Gamma(n)} \int_0^\infty dt\, t^{p-1} e^{-t} E(t)^{n-1} e^{-\theta E(t)} \tag{50}$$

where $E(t) = \int_t^\infty x^{-1}e^{-x}dx$.

The $P_{\alpha,\theta}$ distribution of V_n on $[0,1]$ is not easy to describe explicitly. There is however the following implicit description for $n = 1$:

Proposition 18 *The $P_{\alpha,\theta}$ density of V_1 is uniquely determined for all $0 \leq \alpha < 1$ and $\theta > -\alpha$ by the following identity:*

$$P_{\alpha,\theta}(V_1 \in dx)/dx = \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)} x^{-\alpha-1}(1-x)^{\alpha+\theta-1} P_{\alpha,\alpha+\theta}(V_1 < x/(1-x)) \tag{51}$$

The special case of (51) with $\alpha = 0$ and $\theta = 1$ appears as equation (3) of Vershik [57], attributed to Dickman [13]. See also [63, 24, 43] for alternative approaches to computation of the distribution of V_1 for $\mathrm{PD}(0,\theta)$ and Lamperti [37] for a different functional equation that determines the distribution of $1/V_1$ for $\mathrm{PD}(\alpha,0)$. In Section 8.1 a formula of Perman [43] is applied to obtain an expression for the $P_{\alpha,\theta}$ joint density of $V_1, \cdots, V_n$ for $0 < \alpha < 1, \theta > -\alpha$ which is analogous to known results for $\mathrm{PD}(0,\theta)$ [6, 58, 32]. In particular, this approach yields the following extension of results in Section 4 of [43] for the cases $\theta = 0$ and $\theta = \alpha$. To simplify notation, let $\bar{u} = 1 - u$.

Proposition 19 *For all $0 \leq \alpha < 1$ and $\theta > -\alpha$*

$$P_{\alpha,\theta}(V_1 \in du)/du = \sum_1^\infty (-1)^{n+1} c_{n,\alpha,\theta} \frac{\bar{u}^{\alpha+\theta-1}}{u^{\alpha+1}} I_{n,\alpha,\theta}(u) \qquad (0 < u < 1) \tag{52}$$

where $I_{n,\alpha,\theta}(u) = 0$ if $u > 1/n$, so all but the first n terms of the sum are zero if $u > 1/(n+1)$, $I_{1,\alpha,\theta}(u) = 1$, and for $n = 2, 3, \cdots$ and $0 < u_n \leq 1/n$,

$I_{n,\alpha,\theta}(u_n)$ is the $(n-1)$-fold integral

$$I_{n,\alpha,\theta}(u_n) = \int_{u_n/\bar{u}_n}^{1/(n-1)} du_{n-1}\, \frac{\bar{u}_{n-1}^{2\alpha+\theta-1}}{u_{n-1}^{\alpha+1}} \int_{u_{n-1}/\bar{u}_{n-1}}^{1/(n-2)} du_{n-2}\, \frac{\bar{u}_{n-2}^{3\alpha+\theta-1}}{u_{n-2}^{\alpha+1}} \cdots \int_{u_2/\bar{u}_2}^{1} du_1\, \frac{\bar{u}_1^{n\alpha+\theta-1}}{u_1^{\alpha+1}} \tag{53}$$

and $c_{n,0,\theta} = \theta^n$ while for $0 < \alpha < 1, \theta > -\alpha$

$$c_{n,\alpha,\theta} = \frac{,\,(\theta+1),\,(\frac{\theta}{\alpha}+n)\alpha^{n-1}}{,\,(\theta+n\alpha),\,(\frac{\theta}{\alpha}+1),\,(\bar{\alpha})^n} \tag{54}$$

For $1/2 < u < 1$ there is only one positive term in (52), and the formula reduces to (51). For $1/3 < u \leq 1/2$ there are two non-zero terms in (52). This formula appears in the bridge case $\theta = \alpha$ at the bottom of page 278 of [43], but with a typographical error: 2α, (α) should be replaced by $2\alpha^2$, (α).

To illustrate using Proposition 6, for $\alpha = \theta = 1/2$, Proposition 19 describes the density of the length V_1 of the longest excursion of a Brownian bridge. Explicit integration is possible in this case at least for $n = 1, 2, 3$ to obtain

$$P_{1/2,1/2}(V_1 \in du)/du = q_1(u) - q_2(u) + q_3(u) \text{ for } 1/4 < u < 1 \tag{55}$$

where the $q_n(u)$ are given for $0 < u < 1$ and $n = 1, 2, 3$ by

$$q_1(u) = \frac{1}{2}u^{-3/2} \tag{56}$$

$$q_2(u) = 1(u \leq 1/2)\frac{1}{\pi}u^{-3/2}\left(-\pi + 2\sqrt{\frac{1-2u}{u}} + 2\,\text{arcsine}\sqrt{\frac{u}{1-u}}\right) \tag{57}$$

$$q_3(u) = 1(u \leq 1/3)\frac{3}{4\pi}u^{-3/2}\left(2 + 2\pi + \frac{2}{u} - 8\sqrt{\frac{1-2u}{u}} - 8\,\text{arcsine}\sqrt{\frac{u}{1-u}}\right) \tag{58}$$

See also Wendel [65] for another expression for the $P_{\alpha,\alpha}$ distribution of V_1 based on a method of Rosén, and see Knight [35] for related results.

3.3 A subordinator representation for $0 < \alpha < 1, \theta > 0$

In view of Propositions 4 and 5, it is natural to look for a representation of PD(α, θ) as the distribution of the sequence

$$\left(\frac{V_1(T)}{T}, \frac{V_2(T)}{T}, \cdots\right) \tag{59}$$

derived as in (6) from the ranked lengths $V_n(T)$ of component intervals of the set $[0,T]\backslash Z$, where Z is the closure of the range of a suitable subordinator $(\tau_s, s \geq 0)$, and T is a suitably defined random time. Such a representation is provided by the following Proposition. We write e.g. $\tau(s)$ instead of τ_s when typographically convenient.

Proposition 20 *Fix α with $0 < \alpha < 1$, and $C > 0$. Let $(\tau_s, s \geq 0)$ be a subordinator with Lévy measure $\alpha C x^{-\alpha-1} e^{-x} dx$. Independent of $(\tau_s, s \geq 0)$, let $(\gamma(t), t \geq 0)$ be a gamma subordinator as defined below (7). For $\theta > 0$ let*

$$S_{\alpha,\theta} = \frac{\gamma(\theta/\alpha)}{C, (1-\alpha)} \tag{60}$$

Then for $T = \tau(S_{\alpha,\theta})$ the sequence (59) has $\mathrm{PD}(\alpha,\theta)$ distribution, independently of T, which has the same gamma(θ) distribution as γ_θ.

Notice that in contrast to the formula of Proposition 13, all objects appearing in the above proposition have sensible limits as $\alpha \to 0$ for fixed θ. Take C so that $\alpha C \to 1$ as $\alpha \to 0$. Then as $\alpha \to 0$, the Lévy measure $\alpha C x^{-\alpha-1} e^{-x} dx$ of the subordinator (τ_s) approaches the Lévy measure $x^{-1} e^{-x} dx$ of a gamma process, while $S_{\alpha,\theta}$ converges in probability to the constant θ by the law of large numbers. So in the limit as $\alpha \to 0$ we recover Kingman's representation of $\mathrm{PD}(0,\theta)$ stated in Proposition 4.

Proposition 20 is closely related to the following result, originally obtained by an entirely different argument. See also Proposition 31 below.

Corollary 21 [47] *For $0 < \alpha < 1$ and $\theta > 0$, suppose (U_n) has $\mathrm{PD}(0,\theta)$ distribution, and independent of (U_n) let (V_{mn}), $m = 1, 2 \cdots$ be a sequence of independent copies of (V_n) with $\mathrm{PD}(\alpha, 0)$ distribution. Let (W_n) be defined by ranking the collection of products $\{U_m V_{mn}, m \in \mathbb{N}, n \in \mathbb{N}\}$. Then (W_n) has $\mathrm{PD}(\alpha,\theta)$ distribution.*

4 Development for $\mathrm{PD}(\alpha, 0)$

4.1 Proofs of the main results

We will prove Proposition 9 first, then Proposition 7. Otherwise the proofs are in the same order as the propositions.

Proof of Proposition 9. It is enough to establish the assertions (i), (ii) and (iii) of the Proposition for any particular sequence (V_n) with $\mathrm{PD}(\alpha, 0)$ distribution. So use $V_n := V_n(\tau_1)/\tau_1$ for a stable (α) subordinator (τ_s) as in 12. We first verify a modified form of the assertions (i),(ii) and (iii) in this case, with the definitions (24) replaced by

$$L := C\tau_1^{-\alpha}, \ \Sigma := \tau_1, \ \Delta_n := V_n(\tau_1). \tag{61}$$

The modified form of (ii) follows from the fact that the $V_n(\tau_1)$ are the ranked points of a PRM $\Lambda_\alpha(dx)$ on $(0, \infty)$. The modified form of (iii) follows by the usual change of variables to reduce the inhomogeneous PRM $\Lambda_\alpha(dy)$ on $(0, \infty)$ to a homogeneous PRM dx on $(0, \infty)$. Now (23) with a.s. convergence and $L = C\tau_1^{-\alpha}$ follows because $X_n/n \to 1$ a.s. by the law of large numbers. (This argument is due to Kingman [33]: our formula (23) is his (68)). See Section 4.3 for justification of the convergence (23) in pth mean. Tracing back through these definitions shows that the r.v's defined in (61) can be recovered a.s. from L via (24). Thus (i),(ii) and (iii) hold for any (V_n) with $\mathrm{PD}(\alpha, 0)$ distribution. □

Proof of Proposition 7. By definition of R_n and the notation in Proposition 9,

$$R_n := \frac{V_{n+1}}{V_n} = \frac{\Delta_{n+1}}{\Delta_n} = \left(\frac{X_n}{X_{n+1}}\right)^{\frac{1}{\alpha}} \tag{62}$$

Thus Proposition 7 reduces by a simple change of variables to the following elementary property of the points $0 < X_1 < X_2 < \cdots$ of a homogeneous Poisson process on $(0, \infty)$: the X_n/X_{n+1} are mutually independent beta $(n, 1)$ variables. □

We record for later use the following result, which is easily obtained by examination of the above proof:

Corollary 22 *In the setting of Proposition 9, for each $n = 1, 2, \cdots$ the random vector $(R_1, \ldots, R_n)$ is independent of the random sequence $(X_{n+1}, X_{n+2}, \cdots)$.*

The previous argument shows that for all $\alpha > 0$ formula (35) gives the Laplace transform of A_n defined by the last expression in (30) for a sequence of independent beta$(n\alpha, 1)$ distributed random variables (R_n), or by (64) in terms of Δ_n as in Lemma 23. However, the distribution of Σ_n is of interest only for $0 < \alpha < 1$, as it is easily seen that $\Sigma_n = \infty$ a.s. for $\alpha \geq 1$.

The following Lemma serves as a basis for further calculations.

Lemma 23 *Let* $\Delta_1 > \Delta_2 > \cdots$ *be the ranked points of a* PRM $\Lambda_\alpha(dx)$ *on* $(0,\infty)$, *where* $\Lambda_\alpha(x,\infty) = Cx^{-\alpha}$ *for some* $\alpha > 0$ *and* $C > 0$. *Then*

(i) $C\Delta_n^{-\alpha}$ *has gamma*(n) *distribution;*

(ii) *for* $n \geq 2$ *the* $n-1$ *ratios*

$$\frac{\Delta_1}{\Delta_n} > \frac{\Delta_2}{\Delta_n} > \cdots > \frac{\Delta_{n-1}}{\Delta_n}$$

are distributed like the order statistics of $n-1$ *independent random variables with common distribution* $C^{-1}\Lambda_\alpha(dx)1(x>1)$, *independently of* $\Delta_n, \Delta_{n+1}, \cdots$

(iii) *conditionally given* $\Delta_1, \cdots, \Delta_n$ *for* $n \geq 1$, *the*

$$\frac{\Delta_{n+1}}{\Delta_n} > \frac{\Delta_{n+2}}{\Delta_n} > \cdots$$

are the ranked points of a PRM $\Delta_n^{-\alpha}\Lambda_\alpha(dx)1(x<1)$.

Proof. Basic properties of Poisson processes imply that conditionally given $\Delta_n = a$, for $n \geq 2$ and $a > 0$, the $\Delta_1 > \Delta_2 > \cdots > \Delta_{n-1}$ are distributed like the order statistics of $n-1$ independent random variables with common distribution $\Lambda_\alpha(a,\infty)^{-1}\Lambda_\alpha(dx)1(x>a)$; and conditionally given $\Delta_1, \cdots, \Delta_n$ for $n \geq 1$ with $\Delta_n = a$, the $\Delta_{n+1} > \Delta_{n+2} > \cdots$ are the ranked points of a PRM $\Lambda_\alpha(dx)1(x>a)$. Since under the transformation $u = x/a$ the image of the measure $\Lambda_\alpha(dx)$ is $a^{-\alpha}\Lambda_\alpha(du)$, the assertions of the Lemma follow easily. □

Proof of Proposition 9. Represent the PD$(\alpha,0)$ distributed sequence (V_n) in terms of the points Δ_n of a PRM Λ_α as in Proposition 9. So

$$\frac{1}{V_n} = \frac{\Delta_1 + \cdots + \Delta_{n-1}}{\Delta_n} + \frac{\Delta_n}{\Delta_n} + \frac{\Delta_{n+1} + \Delta_{n+2} + \cdots}{\Delta_n} = A_{n-1} + 1 + \Sigma_n \quad (63)$$

For $n \geq 2$ there is the representation

$$A_{n-1} = \frac{\Delta_1}{\Delta_n} + \frac{\Delta_2}{\Delta_n} + \cdots + \frac{\Delta_{n-1}}{\Delta_n} \quad (64)$$

where the $(\Delta_i/\Delta_n, 1 \leq i \leq n-1)$ are distributed as the ranked values of $n-1$ independent random variables with the same distribution as A_1. Thus

A_n is distributed like the sum of n such independent copies of A_1, which has distribution

$$P(A_1 \in dx) = C^{-1}\Lambda_\alpha(dx)1(x>1) = \alpha x^{-\alpha-1}dx1(x>1) \tag{65}$$

This yields Part (i). Consider now Σ_n defined by (31). Part (iii) of Lemma 23 represents Σ_n conditionally given $\Delta_1, \cdots, \Delta_n$ as the sum of points of a PRM $\Delta_n^{-\alpha}\Lambda_\alpha(dx)1(x<1)$, whence

$$E[\exp(-\lambda\Sigma_n)|\Delta_1,\cdots,\Delta_n] = \exp\left(-\Delta_n^{-\alpha}\int_0^1(1-e^{-\lambda u})\Lambda_\alpha(du)\right) \tag{66}$$

Integration with respect to the gamma(n) distribution of $C\Delta_n^{-\alpha}$ yields (36), which establishes (ii). Finally, the independence claimed in part (iii) follows from the independence assertion in part (ii) of Lemma 23. □

The following conditional form of Wendel's formula (37) proves useful in later calculations:

Proposition 24 *Suppose* (V_n) *has* PD$(\alpha, 0)$ *distribution. Let* (X_n),(R_n), *and* (A_n) *be derived from* (V_n) *as in* (26), (20), *and* (30). *The conditional law of* V_n *given* $R_1, \cdots, R_{n-1}$ *and* X_n *is characterized by*

$$E\left[\exp\left(-\frac{\lambda}{V_n}\right) \mid R_1,\cdots,R_{n-1},X_n\right] = \exp(-\lambda(1+A_{n-1}))\exp[-X_n(\psi_\alpha(\lambda)-1)] \tag{67}$$

Proof. Represent (V_n) in terms of the points (Δ_n) of a P.R.M. Λ_α as in (25). Note that $\sigma(R_1,\cdots,R_{n-1},X_n) = \sigma(\Delta_1,\cdots,\Delta_n)$ and use (34), (66), and $\Delta_n^{-\alpha} = X_n/C$. □

Consider now H_n derived as in (38) from the range of a stable (α) subordinator. Note that at time H_n the nth longest excursion interval that currently has length 1 is necessarily the meander interval. That is to say: $G_{H_n} = H_n - 1$, where for $t \geq 0$ we set

$$G_t = \sup(Z\cap[0,t)); \qquad D_t = \inf(Z\cap[t,\infty)) \tag{68}$$

Notice that H_n is just the nth instant t such that $t - G_t = 1$, so

$$0 < G_{H_1} < D_{H_1} < G_{H_2} < D_{H_2} < \cdots < G_{H_{n-1}} < D_{H_n} < G_{H_{n+1}}$$

and there is the natural decomposition

$$H_n = \sum_{j=1}^{n}(G_{H_j} - D_{H_{j-1}}) + \sum_{j=1}^{n-1}(D_{H_j} - G_{H_j}) + (H_n - G_{H_n}) \qquad (69)$$

where $D_{H_0} = 0$ by convention, and the last term is $H_n - G_{H_n} = 1$. As shown by Wendel, formula (37) follows from the identity in distribution (39) and the observation that the first sum on the right side of (69) is a sum of n independent terms with

$$G_{H_j} - D_{H_{j-1}} \stackrel{d}{=} G_{H_1} \stackrel{d}{=} \Sigma_1 \quad (1 \leq j \leq n) \qquad (70)$$

while the $n-1$ terms of the second sum are independent with

$$D_{H_j} - G_{H_j} \stackrel{d}{=} A_1 \quad (1 \leq j \leq n-1) \qquad (71)$$

where Σ_1 and A_1 are as in Proposition 10. These observations can be checked by repeated application of the strong Markov property at the times H_{D_j}, and the Poisson character of excursion interval lengths. Note that the $V_j(H_n)$ for $1 \leq j \leq n-1$ are the ranked values of the i.i.d. interval lengths $D_{H_j} - G_{H_j}, 1 \leq j \leq n-1$, while $V_n(H_n) = 1$.

Proof of Proposition 12. The Poisson character of the interval lengths on the local time scale implies that for each fixed n the distribution of $(V_m(H_n), m = 1, 2, \cdots)$ can be described as follows:

(i) $V_n(H_n) = 1$;

(ii) for $0 < m < n$ the $V_m(H_n)$ are distributed like the order statistics of $m-1$ independent r.v's with distribution $C^{-1}\Lambda_\alpha(dx)1(x > 1)$;

(iii) independent of the $V_m(H_n)$ for $0 < m < n$, the multiple of the local time CS_{H_n} has a gamma(n) distribution;

(iv) given S_{H_n} and the $V_m(H_n)$ for $0 < m < n$, the $V_m(H_n)$ for $n < m < \infty$ are distributed as the ranked points of a PRM $S_{H_n}\Lambda_\alpha(dx)1(x < 1)$.

On the other hand, Lemma 23 shows that the same four statements hold if the following substitutions are made:

replace $V_m(H_n)$ by Δ_m/Δ_n and replace S_{H_n} by $\Delta_n^{-\alpha}$

where the Δ_n are the ranked points of a PRM $\Lambda_\alpha(dx)$. Therefore, for each fixed $n = 1, 2, \cdots$,

$$\left(\frac{V_m(H_n)}{V_n(H_n)}, m = 1, 2, \cdots\right) \stackrel{d}{=} \left(\frac{\Delta_m}{\Delta_n}, m = 1, 2, \cdots\right) \tag{72}$$

The distribution of the sequence in (40) is now identified as PD$(\alpha, 0)$ using Proposition 9 (ii). □

4.2 A differential equation related to ϕ_α and ψ_α.

A proof of (35) can also be obtained using the recurrence relation

$$A_n = (1 + A_{n-1})/R_n \tag{73}$$

and the fact that

$$e^\lambda \phi_\alpha(\lambda) = E\left[\exp -\lambda\left(\frac{1}{R_1} - 1\right)\right] \tag{74}$$

solves the differential equation

$$\alpha = (\alpha + \lambda) f(\lambda) - \lambda f'(\lambda). \tag{75}$$

Another solution of (75) is the function

$$e^\lambda \psi_\alpha(\lambda) = \left(E\left[\exp -\lambda\left(\frac{1}{V_1}\right)\right]\right)^{-1} \tag{76}$$

In fact, all solutions of (75) are given by the formula

$$f(\lambda) = \lambda^\alpha e^\lambda \left[c + \alpha \int_\lambda^\infty \frac{dx e^{-x}}{x^{\alpha+1}}\right] \tag{77}$$

where $c = \lim_{\lambda\to\infty} \lambda^{-\alpha} e^{-\lambda} f(\lambda)$ is an arbitrary constant. Hence, $e^\lambda \phi_\alpha(\lambda)$ is the solution of (75) with $c = 0$, whereas $e^\lambda \psi_\alpha(\lambda)$ is the solution of (75) with $c =$, $(1 - \alpha)$, in agreement with formula (33). It can also be checked that the fact that $e^\lambda \psi_\alpha(\lambda)$ solves (75), together with the recurrence

$$\Sigma_n = R_n(1 + \Sigma_{n+1}), \tag{78}$$

is in agreement with formula (36). But, in contrast with the situation for (35), it seems difficult to prove (36) from this approach.

4.3 Some Absolute Continuity Relationships

For (X_n) the points of a homogeneous Poisson process on $(0, \infty)$ with rate 1 there is the elementary absolute continuity relation

$$E[f(X_{m+1}, X_{m+2}, \cdots)] = \frac{1}{m!} E[X_1^m f(X_1, X_2, \cdots)], \tag{79}$$

where f is a generic positive measurable function of its arguments. For R_n as in (62), a change of variables yields

$$E[f(R_{m+1}, R_{m+2}, \cdots)] = \frac{1}{m!} E[X_1^m f(R_1, R_2, \cdots)], \tag{80}$$

where by a paraphrase of (23)

$$X_1 = \lim_{n \to \infty} n(R_1 R_2 \cdots R_n)^\alpha \text{ a.s.} \tag{81}$$

On the other hand, a direct calculation of the density ratio using Proposition 7 shows that

$$E[f(R_{m+1}, R_{m+2}, \cdots, R_{m+n})] = \binom{n+m}{m} E[(R_1 R_2 \cdots R_n)^{m\alpha} f(R_1, R_2, \cdots, R_n)] \tag{82}$$

Comparison of (80) and (82) shows that

$$E\left(\frac{X_1^m}{m!} | R_1, \ldots, R_n\right) = \binom{n+m}{m} (R_1 R_2 \cdots R_n)^{m\alpha} \tag{83}$$

Since X_1 has finite moments of all orders, martingale convergence shows that the a.s. convergence in (81) takes place also in pth mean for every $p > 0$. It follows easily that the same is true of the a.s. convergence in (23).

5 Development for PD(α, θ)

5.1 Proofs of some results

Proof of Proposition 16. Combine Proposition 13 and the following Lemma. □

Lemma 25 *Suppose* (V_n) *has* PD$(\alpha, 0)$ *distribution, and let* $L = \lim_n nV_n^\alpha$ *as in* (23). *Then for all real* $\ell > -1$ *and* $p > 0$, *and* $n = 1, 2 \cdots$

$$E[L^\ell\, V_n^p] = \frac{\Gamma(\ell+n)}{\Gamma(n)\Gamma(p+\ell\alpha)} \int_0^\infty dt\, t^{p+\ell\alpha-1} e^{-t} \phi_\alpha(t)^{n-1} \psi_\alpha(t)^{-\ell-n} \tag{84}$$

Proof. We will use the following standard expression for negative moments of a positive random variable X in terms of its Laplace transform: for $p > 0$,

$$E[\, X^{-p}\,] = \frac{1}{\Gamma(p)} \int_0^\infty dt\, t^{p-1} E[\, e^{-tX}\,], \tag{85}$$

Combined with Wendel's formula (37), this immediately yields the special case of (84) with $\ell = 0$. Recall that $X_n := LV_n^{-\alpha}$. Then the left side of (84) is

$$E[L^\ell\, V_n^p] = E(X_n^\ell\, V_n^{p+\ell\alpha}) = E\left[X_n^\ell \frac{1}{\Gamma(p+\ell\alpha)} \int_0^\infty dt\, t^{p+\ell\alpha-1} E[\exp(-t/V_n) \mid X_n]\right]$$

by using (85). Now use (67), and the fact that X_n has gamma (n) distribution independent of A_{n-1} to obtain by elementary integration

$$E[L^\ell\, V_n^p] = \frac{1}{\Gamma(n)\Gamma(p+\ell\alpha)} \int_0^\infty dt\, t^{p+\ell\alpha-1} e^{-t}\Gamma(\ell{+}n)\psi_\alpha(t)^{-\ell-n} E[\exp(-tA_{n-1})]$$

Here, for $n = 1$, $A_0 = 0$. Now use (35) to obtain (84). □

Remark 26 Consider (84) for $p = 0, \ell > 0$. Since the left side does not depend on n, neither does the right, something which is not evident a priori. This can be shown to be equivalent to the Wronskian identity

$$(\phi_\alpha \psi'_\alpha - \psi_\alpha \phi'_\alpha)(t) = \alpha\Gamma(1-\alpha) e^{-t} t^{\alpha-1} \tag{86}$$

which follows from the description of ϕ_α and ψ_α in terms of the differential equation (75).

Further moment formulae. Suppose (V_n) has PD$(\alpha, 0)$ distribution. Let L, X_n, R_n, A_n and Σ_n be the random variables defined in terms of (V_n) as in (23), (26), (20), (30), and (31).

As a first variant of (84), we can compute similarly

$$E[L^\ell\, V_n^p \exp(-\lambda/V_n)] = E[X_n^\ell\, V_n^{p+\ell\alpha} \exp(-\lambda/V_n)]$$

$$= E[X_n^\ell\, \Gamma(p+\ell\alpha)^{-1} \int_0^\infty dt\, t^{p+\ell\alpha-1} E[\,\exp(-(t+\lambda)/V_n|X_n\,]$$

Using (67) and then (35) again, with $t+\lambda$ instead of λ, yields

$$E[L^\ell\, V_n^p \exp(-\lambda/V_n)] = \frac{\Gamma(\ell+n)}{\Gamma(n)\Gamma(p+\ell\alpha)} \int_0^\infty dt\, t^{p+\ell\alpha-1} e^{-t-\lambda} \phi_\alpha(t+\lambda)^{n-1} \psi_\alpha(t+\lambda)^{-\ell-n} \tag{87}$$

Proof of Proposition 15. This follows easily from Propositions 7 and 13 using the following Lemma, which states another variant of (84):

Lemma 27 *Suppose* (V_n) *has* PD$(\alpha,0)$ *distribution. Let* $L := \lim_{n\to\infty} nV_n^\alpha$ *as in* (23)*, and let* $R_n := V_{n+1}/V_n$*. For all real* $\ell > -1$ *and* $\gamma > 0$*, and* $n = 0,1,2\cdots$

$$E[L^\ell\, V_1^{\gamma-\alpha\ell} \mid R_1,\ldots,R_n] = \frac{1}{n!}\left(\prod_{j=1}^n R_j\right)^{\ell\alpha-\gamma} \Phi_\alpha(\ell+n,\gamma,A_n) \tag{88}$$

for A_n *as in* (30) *and* $\Phi_\alpha(\ell,\gamma,a)$ *as in* (47).

Proof. Let $\mathcal{R}_n = \sigma(R_1,\ldots,R_n)$. Elementary manipulations show that

$$E[L^\ell\, V_1^{\gamma-\alpha\ell}\, |\mathcal{R}_n] = \left(\prod_{j=1}^n R_j\right)^{\ell\alpha-\gamma} E[(L/V_{n+1}^\alpha)^\ell\, V_{n+1}^\gamma \mid \mathcal{R}_n] \tag{89}$$

Now use (85) for $p=\gamma$ and $X = 1/V_{n+1}$ to express the right side of (89) as:

$$\left(\prod_{j=1}^n R_j\right)^{\ell\alpha-\gamma} \frac{1}{\Gamma(\gamma)} \int_0^\infty dt\, t^{\gamma-1}\{\cdots\}$$

where

$$\{\cdots\} = E[(L/V_{n+1}^\alpha)^\ell\, E[\exp(-t/V_{n+1}) \mid \mathcal{R}_n, X_{n+1}] \mid \mathcal{R}_n]$$

and $X_{n+1} := LV_{n+1}^{-\alpha}$. Now use formula (67) to show that

$$\{\cdots\} = E[X_{n+1}^\ell\, \exp(X_{n+1}(1-\psi_\alpha(t))]\exp(-t(1+A_n))$$

$$= \exp(-t(1+A_n))\,\frac{\Gamma(n+\ell)}{n!}\,(\psi_\alpha(t))^{-(\ell+n+1)}$$

by the independence of A_n and X_{n+1} (see Corollary 22 and (30)) and elementary integration with respect to the gamma$(n+1)$ distribution of X_{n+1}. This yields formula (88) with Φ_α defined by (47). The second equality in (47) is easily obtained by another manipulation like (85). □

Remark 28 It is also possible to derive (88), with Φ_α defined by the second expression in (47), by starting from Perman's formula for the joint density of $\Sigma, V_1, \cdots, V_{n+1}$ stated in Proposition 45, and making suitable changes of variables and integrating out Σ and V_1.

Proof of Proposition 18 For $(\tilde{V}_n)$ the size-biased permutation of (V_n) as in Definition 1 and Proposition 2, we can compute $P_{\alpha,\theta}(V_1 \in dx, V_1 = \tilde{V}_1)$ in two different ways. First, by conditioning on V_1 and using (2),

$$P_{\alpha,\theta}(V_1 \in dx, V_1 = \tilde{V}_1) = x P_{\alpha,\theta}(V_1 \in dx) \tag{90}$$

But conditioning instead on $\tilde{V}_1$, and using the consequence of (4) that the $P_{\alpha,\theta}$ distribution of $(\tilde{V}_2, \tilde{V}_3, \cdots)/(1-\tilde{V}_1)$ is identical to the $P_{\alpha,\alpha+\theta}$ distribution of $(\tilde{V}_1, \tilde{V}_2, \cdots)$ yields

$$P_{\alpha,\theta}(V_1 \in dx, V_1 = \tilde{V}_1) = P_{\alpha,\theta}(\tilde{V}_1 \in dx, \max_{n\geq 2} \tilde{V}_n < x)$$

$$= P_{\alpha,\theta}(\tilde{V}_1 \in dx) P_{\alpha,\theta}(\max_{n\geq 2} \frac{\tilde{V}_n}{1-\tilde{V}_1} < \frac{x}{1-x} \,|\tilde{V}_1 = x)$$

$$= \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)\Gamma(1-\alpha)} x^{-\alpha}(1-x)^{\alpha+\theta-1} dx P_{\alpha,\alpha+\theta}(V_1 < x/(1-x)) \tag{91}$$

Now comparison of (90) and (91) yields (51). For $1/2 < x < 1$ it is obvious that $P_{\alpha,\alpha+\theta}(V_1 < x/(1-x)) = 1$, so (51) determines the $P_{\alpha,\theta}$ density of V_1 at x for $1/2 < x < 1$. (This case of (51) can also be read from (5)). Recursive application of (51) now determines the $P_{\alpha,\theta}$ density of V_1 at x for $1/(n+1) < x < 1/n$, $n = 2, 3, \cdots$. □

Proof of Proposition 20. Let $K = C\Gamma(1-\alpha)$. Let (σ_s) be a stable(α) subordinator with $E[\exp(-\lambda\sigma_s)] = \exp(-K\lambda^\alpha s)$. Then for each $s > 0$ and every positive measurable functional F,

$$E[\,F(\tau_t, 0 \leq t \leq s)\,] = e^{Ks} E[\,F(\sigma_t, 0 \leq t \leq s) \exp(-\sigma_s)\,] \tag{92}$$

Let $(V_1, V_2, \cdots)$ denote a sequence with $\text{PD}(\alpha, \theta)$ distribution. Let L be the local time variable derived from $(V_1, V_2, \cdots)$ as in (23), and $\Sigma = (C/L)^{1/\alpha}$. From Propositions 13 and 9, the conditional law of $(V_1, V_2, \cdots)$ given $\Sigma = t$ does not depend on θ, call it $\text{PD}(\alpha | t)$ say:

$$\text{PD}(\alpha | t) = \text{the conditional law of } \left(\frac{\Delta_1}{\sigma_1}, \frac{\Delta_2}{\sigma_1}, \cdots\right) \text{ given } \sigma_1 = t \tag{93}$$

where $\Delta_1 > \Delta_2 > \cdots$ are the ranked jumps of $(\sigma_s, 0 \leq s \leq 1)$. Then from (43)

$$\text{PD}(\alpha, \theta) = c_{\alpha,\theta} \int_0^\infty \text{PD}(\alpha | t) t^{-\theta} P(\sigma_1 \in dt). \tag{94}$$

The finite dimensional distributions of $\text{PD}(\alpha | t)$ are described by Perman's formula (152), but this description is not required in the following argument.

Let

$$\mathbf{W}_s = \left(\frac{V_1(\tau_s)}{\tau_s}, \frac{V_2(\tau_s)}{\tau_s}, \cdots\right)$$

From (92) we learn that if ζ is a positive random variable independent of $(\tau_s, s \geq 0)$, then

$$\textit{the conditional law of } \mathbf{W}_\zeta \textit{ given } \zeta \textit{ and } \tau_\zeta \textit{ is } \text{PD}(\alpha | \tau_\zeta / \zeta^{1/\alpha}) \tag{95}$$

no matter what the distribution of ζ. Consequently

$$\zeta \textit{ and } \mathbf{W}_\zeta \textit{ are conditionally independent given } \tau_\zeta/\zeta^{1/\alpha}. \tag{96}$$

From (94) and (96), it now suffices to show that for $\zeta = K^{-1}\gamma(\theta/\alpha)$ the following three things are true:

$$P[\tau_\zeta/\zeta^{1/\alpha} \in dt] = c_{\alpha,\theta} t^{-\theta} P(\sigma_1 \in dt) \tag{97}$$

$$\tau_\zeta \textit{ has gamma}(\theta) \textit{ distribution} \tag{98}$$

$$\tau_\zeta/\zeta^{1/\alpha} \textit{ and } \tau_\zeta \textit{ are independent;} \tag{99}$$

But (97), (98) and (99) follow at once from the next Lemma applied with $h(z) = cz^b$ for $b = (\theta/\alpha) - 1$ and a constant c.

Lemma 29 *Let* $(\tau_s, s \geq 0)$ *be as in Proposition* 20 *and let* ζ *be a random variable independent of* $(\tau_s, s \geq 0)$ *with density of the form*

$$P(\zeta \in dz)/dz = h(z)\exp(-Kz) \tag{100}$$

for some function $h(z)$. *Then for* $t > 0, u > 0$

$$P\left(\tau_\zeta \in du,\ \frac{\tau_\zeta}{\zeta^{1/\alpha}} \in dt\right) = \alpha e^{-u}\frac{u^{\alpha-1}}{t^\alpha} h\left(\left(\frac{u}{t}\right)^\alpha\right) du\, P(\sigma_1 \in dt) \tag{101}$$

Proof. Conditioning on $\zeta = z$, there is the following identity for all positive measurable functions f and g:

$$\begin{aligned} E[\,f(\tau_\zeta)g(\tau_\zeta/\zeta^{1/\alpha})\,|\,\zeta = z\,] &= E[\,f(\tau_z)g(\tau_z/z^{1/\alpha})\,] \\ &= \exp(Kz)E[\,f(z^{1/\alpha}\sigma_1)g(\sigma_1)\exp(-z^{1/\alpha}\sigma_1)\,] \end{aligned}$$

by (92) and the scaling property of the stable subordinator $(\sigma_s, s \geq 0)$. Integrate with respect to the distribution (100) of ζ to obtain

$$\begin{aligned} E[\,f(\tau_\zeta)g(\tau_\zeta/\zeta^{1/\alpha})] &= \int_0^\infty dz\, h(z)E[\,f(z^{1/\alpha}\sigma_1)g(\sigma_1)\exp(-z^{1/\alpha}\sigma_1)\,] \\ &= E\left[\int_0^\infty du\,\alpha e^{-u}\frac{u^{\alpha-1}}{\sigma_1^\alpha} h\left(\left(\frac{u}{\sigma_1}\right)^\alpha\right) f(u)g(\sigma_1)\right] \end{aligned} \tag{102}$$

by Fubini's theorem and the change of variable

$$u = z^{1/\alpha}\sigma_1, \qquad z = (u/\sigma_1)^\alpha, \qquad dz = \alpha(u^{\alpha-1}/\sigma^\alpha)du.$$

Now (102) amounts to (101). □

Remark 30 Conversely, formula (101) shows that if any of (97), (98) or (99) holds, the function $h(z)$ introduced in (100) must be of the form $h(z) = cz^b$, that is $K\zeta$ must have gamma(b) distribution for some $b > 0$. Consider for instance (99). From (101), for (99) to be satisfied, it is necessary that

$$h(u/v) = j(u)k(v), \qquad du\,dv \text{a.e.}$$

for some functions j and k, hence that

$$h(uw) = c\,h(u)h(w), \qquad du\,dw \text{a.e.}$$

which forces $h(u) = cu^b$ for some c and b.

Proof of Proposition 21. Proposition 21 follows from Proposition 20 and the next Proposition, which in fact allows either of Propositions 21 or 20 to be derived easily from the other.

Proposition 31 *In the setting of Proposition 20, let* $\zeta_t = K^{-1}\gamma(t)$*, where* $K = C, (1-\alpha)$*, and let* $S_1 > S_2 > \cdots$ *denote the ranked values of the jumps of* $(\zeta_t, 0 \leq t \leq \theta/\alpha)$*, say* $S_i = \zeta_{\tau_i} - \zeta_{\tau_i -}$ *where* τ_i *is the time of jump of magnitude* S_i*. Let* $T_i = \tau(\zeta_{\tau_i}) - \tau(\zeta_{\tau_i -})$ *Then*

(i) the $(S_i, T_i), i = 1, 2 \cdots$ *are the points of a* PRM *with intensity measure*

$$M(ds, dt) = \frac{\theta}{\alpha}\frac{ds}{s} f_s(t) e^{-t} dt = \theta \frac{dt}{t} e^{-t} g_t(s) ds \tag{103}$$

where $f_s(t) = P(\sigma_s \in dt)/dt$ *and* $g_t(s) = P(S_t \in ds)/ds$ *where* $(S_t, t \geq 0)$ *is the inverse of the stable* (α) *subordinator* $(\sigma_s, s \geq 0)$*.*

(ii) Let $T_{\pi(i)}$ *be the ith largest of the jumps* $T_i, i = 1, 2, \cdots$*. Then*

$$\left(\frac{T_{\pi(i)}}{\tau(\zeta_{\theta/\alpha})} \ , i = 1, 2, \ldots \right) \textit{has } \mathrm{PD}(0, \theta) \textit{ distribution}$$

independently of the gamma(θ) *variable* $\sum_i T_i = \tau(\zeta_{\theta/\alpha})$

(iii) if $\Delta_{i1} > \Delta_{i2} > \cdots$ *are the ranked jumps of* (τ_s) *incurred over the s-interval whose length is* $S_{\pi(i)}$*, then for each i the sequence*

$$\left(\frac{\Delta_{ij}}{T_{\pi(i)}}, j = 1, 2, \ldots \right) \ \textit{has } \mathrm{PD}(\alpha, 0) \textit{ distribution}$$

Moreover these sequences are mutually independent as i varies, and independent also of the sequence $(T_{\pi(i)}, i = 1, 2, \ldots)$*, where*

$$T_{\pi(i)} = \Delta_{i1} + \Delta_{i2} + \cdots \ \textit{ and } \tau(\zeta_{\theta/\alpha}) = \sum_i T_{\pi(i)} = \sum_i \sum_j \Delta_{ij}$$

and the $V_n(\zeta_{\theta/\alpha})$ *featured in Proposition 20 are the ranked values of the* Δ_{ij}*.*

Proof. Due to the Poisson character of the jumps of the two independent subordinators, the points $(S_i, T_i), i = 1, 2 \cdots$ are the points of a PRM with intensity measure

$$M(ds, dt) = \frac{\theta}{\alpha}\frac{ds}{s} \exp(-Ks) P(\tau_s \in dt) \tag{104}$$

which can be expressed as in (103) using (92) and the formula $f_s(t) = \alpha s g_t(s)/t$ which is a consequence of the identity in distribution $S_t/t^\alpha \stackrel{d}{=} s/\sigma_s^\alpha$ (see e.g. Section 7 of Pitman-Yor [50]). This yields (i). Since $\int_0^\infty g_t(s)ds = 1$, the T_i are the points of a PRM $\theta t^{-1}e^{-t}dt$ over $t > 0$. So (ii) follows from Proposition 4. Turning to (iii), the last expression for $M(ds, dt)$ in (104), combined with standard facts about Poisson processes, shows that conditionally given all the $T_{\pi(i)}$, the corresponding jumps $S_{\pi(i)}$ of the gamma process $(\zeta_t, 0 \le t \le \theta/\alpha)$ are mutually independent, with

$$P(S_{\pi(i)} \in ds \,|\, T_{\pi(i)} = t) = g_t(s)ds$$

Now (iii) follows using (95) and (94) for $\theta = 0$. □

5.2 Limits as $\alpha \to 0$

Let $\mathcal{P}$ denote the space of probability measures on $[0,1] \times [0,1] \times \cdots$, and give $\mathcal{P}$ the topology of weak convergence of finite dimensional distributions. It is immediate from Definition 1 that the $P_{\alpha,\theta}$ distribution of $(\tilde{V}_n)$ defines a continuous map from $\{(\alpha,\theta) : 0 \le \alpha < 1, \theta > -\alpha\}$ to $\mathcal{P}$. As a consequence [16], the same is true of the $P_{\alpha,\theta}$ distribution of (V_n). That is to say, PD(α,θ) is continuous in (α,θ). In particular, for each $\theta > 0$ the limit of PD(α,θ) as $\alpha \downarrow 0$ is PD$(0,\theta)$. That is, for every bounded continuous function f defined on $[0,1]^n$,

$$\lim_{\alpha \downarrow 0} E_{\alpha,\theta}[f(V_1,\cdots,V_n)] = E_{0,\theta}[f(V_1,\cdots,V_n)] \tag{105}$$

Proposition 20 provides a setting in which (105) follows from weak convergence as $\alpha \downarrow 0$ of a subordinator with Lévy measure $x^{-\alpha-1}e^{-x}dx$ to a gamma process with Lévy measure $x^{-1}e^{-x}dx$. See Vershik-Yor [56] for further discussion, and Brockwell-Brown [9] for other aspects of the asymptotic behaviour of a stable α subordinator as $\alpha \downarrow 0$.

To illustrate (105), we now derive the known formula for $E_{0,\theta}(V_n^p)$ for $p > 0$ given in Corollary 17 from the corresponding formula for $E_{\alpha,\theta}(V_n^p)$ with $0 < \alpha < 1$ stated in Proposition 16.

Derivation of Corollary 17 from Proposition 16. The evaluation of the limit is justified by the following asymptotics as $\alpha \to 0$:

$$,\ (1-\alpha)^{\frac{\theta}{\alpha}} \sim (1+\gamma\alpha)^{\frac{\theta}{\alpha}} \to e^{-\theta\gamma} \tag{106}$$

where $a(\alpha) \sim b(\alpha)$ means $a(\alpha)/b(\alpha) \to 1$,

$$\gamma = -\Gamma'(1) \tag{107}$$

is Euler's constant, and

$$\frac{\Gamma(\frac{\theta}{\alpha}+n)}{\Gamma(\frac{\theta}{\alpha}+1)} \sim \frac{\theta^{n-1}}{\alpha^{n-1}} \tag{108}$$

The factor of α^{n-1} in the denominator is asymptotically cancelled inside the integral, by the factor

$$\phi_\alpha(t)^{n-1} = \left(\alpha t^\alpha \int_t^\infty dx\, x^{-\alpha-1} e^{-x}\right)^{n-1} \sim \alpha^{n-1} E(t)^{n-1} \tag{109}$$

Finally, in view of (107) and (109) for $n = 2$, formula (33) implies

$$\psi_\alpha(t) - 1 \sim \alpha(E(t) + \gamma + \log(t)) \tag{110}$$

and consequently

$$\psi_\alpha(t)^{-n-\frac{\theta}{\alpha}} \to e^{-E(t)-\gamma-\log(t)} = t^{-\theta} e^{-\gamma} e^{-E(t)} \tag{111}$$

It is easily argued that these limiting operations can be switched with the integral in (48), and (50) results after some cancellation. □

6 Sampling from $\text{PD}(\alpha, \theta)$

Applications of a random discrete distribution (V_n) often involve a *sample from* (V_n) that is a random variable N such that the conditional distribution of N given (V_n) is given by

$$P(N = n | V_1, V_2, \cdots) = V_n \qquad (n = 1, 2, \cdots) \tag{112}$$

Then V_N is a size-biased pick from (V_n), as in (2). See for instance [64, 23] for a nice interpretation of N in the application of $\text{PD}(0, \theta)$ to population genetics.

6.1 Deletion and insertion operations

Given a sequence (v_n) and an index N, say (v'_n) is derived from (v_n) by *deletion of* v_N if

$$v'_n = v_n 1(n < N) + v_{n+1} 1(n \geq N).$$

The following Proposition follows immediately from Proposition 2:

Proposition 32 *Let N be a sample from (V_n) with* $\text{PD}(\alpha, \theta)$ *distribution, where $0 \leq \alpha < 1$ and $\theta > -\alpha$. Let (V'_n) be derived from (V_n) by deletion of V_N, and let $V''_n = V'_n/(1 - V_N), n = 1, 2, \cdots$. Then (V'_n) has* $\text{PD}(\alpha, \theta + \alpha)$ *distribution, independently of V_N, which has beta$(1 - \alpha, \theta + \alpha)$ distribution.*

In particular the $\text{PD}(0, \theta)$ distribution is invariant under this operation of size-biased deletion and renormalization, a result which is a known characterization of $\text{PD}(0, \theta)$ [41, 27].

Suppose a $\text{PD}(\alpha, 0)$ distributed sequence (V_n) has been constructed by any of the methods described in Section 2. By the operation of size-biased deletion and renormalization as above we obtain a sequence with $\text{PD}(\alpha, \alpha)$ distribution. Repeating the operation yields sequences with distributions $\text{PD}(\alpha, 2\alpha)$, $\text{PD}(\alpha, 3\alpha), \cdots$.

This result about deletion can be rephrased as a result about insertion: given $(v'_1 \geq v'_2 \geq \cdots)$ and a real number $v > \inf_n v'_n$, say (v_n) is derived from (v'_n) by *insertion of* v if

$$v_n = v'_n 1(n < N) + v 1(n = N) + v'_{n+1} 1(n > N).$$

where $N - 1 = \sum_{n=1}^{\infty} 1(v'_n > v)$ is the number of terms of (v'_n) that strictly exceed v. Note that $v_N = v$ by definition.

Proposition 33 *Fix $0 \leq \alpha < 1$ and $\theta > -\alpha$. Let (V''_n) have* $\text{PD}(\alpha, \alpha + \theta)$ *distribution. Independent of (V''_n) let X have beta$(1 - \alpha, \theta + \alpha)$ distribution. Let (V_n) be defined by insertion of X into $((1 - X)V'_n, n = 1, 2, \cdots)$. Then (V_n) has* $\text{PD}(\alpha, \theta)$ *distribution, and $X = V_N$ where N is a sample from (V_n).*

6.2 Distribution of a sample from $\text{PD}(\alpha, \theta)$

Immediately from (111), the unconditional distribution of a sample N from (V_n) with $\text{PD}(\alpha, \theta)$ distribution is given by $P_{\alpha,\theta}(N = n) = E_{\alpha,\theta}(V_n)$ as specified in formulae (48) and (50) for $p = 1$. For $\text{PD}(0, \theta)$ this result is due to

Griffiths [23]. Inspection of formula (50) for $p = 1$ shows that Griffiths' result can be restated as follows:

> *for N a sample from* PD$(0, \theta)$ *, the distribution of $N-1$ is a mixture of Poisson (μ) distributions, with the parameter μ given the distribution of $\theta\Lambda(T, \infty)$, where $\Lambda(dx) = x^{-1}e^{-x}dx$ is the Lévy measure of a gamma subordinator, and T is a standard exponential variable.*

This result can be understood probabilistically as follows, by application of Propositions 4 and 33. Take (V_n'') in Proposition 33 to be the PD$(0, \theta)$ sequence $V_n'' = V_n(\tau_\theta)/\tau_\theta$ derived from a gamma subordinator $(\tau_s, 0 \leq s \leq \theta)$ as in (9). Let $X = T/(T + \tau_\theta)$ for T a standard exponential independent of (τ_s), and let (V_n) be constructed as in Proposition 33. Let N be the rank of X in (V_n). According to Proposition 33, N is a sample from the PD$(0, \theta)$ sequence (V_n). But by construction, $N - 1$ is the number of n such that $V_n(\tau_\theta) > T$, and given T this number has Poisson distribution with mean $\theta\Lambda(T, \infty)$.

The analog for $0 < \alpha < 1$ of the above result for PD$(0, \theta)$ is the subject of the next proposition:

Proposition 34 *For each $0 < \alpha < 1, \theta > -\alpha$ the $P_{\alpha,\theta}$ distribution of $N-1$ is an integral mixture of negative binomial distributions with parameters $\frac{\theta}{\alpha} + 1$ and p, with a mixing distribution over p which depends only on α. More precisely, for each $m = 0, 1, \cdots$*

$$P_{\alpha,\theta}(N-1 = m) = E_{\alpha,\theta}(V_{m+1}) = \int_0^\infty P(Z_{1-\alpha} \in dz) \binom{\frac{\theta}{\alpha} + m}{m} (1-p_\alpha(z))^m p_\alpha(z)^{\frac{\theta}{\alpha}+1} \tag{113}$$

where $Z_{1-\alpha}$ has gamma$(1 - \alpha)$ distribution, and

$$p_\alpha(z) = \frac{\psi_\alpha(z) - \phi_\alpha(z)}{\psi_\alpha(z)} = \frac{\Gamma(1-\alpha)z^\alpha}{\psi_\alpha(z)} \tag{114}$$

is such that $0 < p_\alpha(z) < 1$ for all $0 < \alpha < 1$ and $z > 0$.

Proof. This can be obtained either by manipulation of formula (48) for $p = 1$, or more probabilistically by application of Proposition 33, as in the case $\alpha = 0$ discussed above, using the construction of Proposition 20 instead of Proposition 4. □

From (113) and the formula $r(1-p)/p$ for the mean of the negative binomial (r,p) distribution, for $0<\alpha<1, \theta>-\alpha$ there is the following formula for the mean of N:

$$E_{\alpha,\theta}(N) = 1 + (1+\tfrac{\theta}{\alpha}),\, (1-\alpha)^{-2}\int_0^\infty dz\, z^{-2\alpha} e^{-z}\phi_\alpha(z) \qquad (115)$$

which is linear in θ for fixed $\alpha < \frac{1}{2}$, and infinite for all $\theta > -\alpha$ if $\alpha \geq \frac{1}{2}$. Formulae for higher moments of N follow similarly, while asymptotics for $P_{\alpha,\theta}(N=n)$ and $P_{\alpha,\theta}(N \geq n)$ for large n are immediate from (49).

The next two subsections illustrate two interesting special cases of Proposition 34 with natural interpretations in terms of excursions of a Brownian motion or Bessel process. We thank Yuval Peres and Steve Evans for a conversation which helped us develop these interpretations.

6.3 The rank of the excursion in progress

Consider the set up of Section 1.2, with Z the range of a stable(α) subordinator, and $V_n(t)$ the length of the nth longest interval component of $[0,t]\backslash Z$. So Z could be the zero set of Brownian motion ($\alpha = \frac{1}{2}$), or a recurrent Bessel process of dimension $2-2\alpha$ for $0<\alpha<1$. Let N_t be the rank of the meander length $t-G_t$ in the sequence of excursion lengths $V_1(t) > V_2(t) > \cdots$, so $t - G_t = V_{N_t}(t)$. According to Theorem 1.2 of [50], for each fixed time t the random variable N_t is a sample from $(V_n(t)/t)$. Combined with (13), this shows that the joint law of N_t and the sequence $(V_n(t)/t)$ is given by the formula

$$E\left[1(N_t = n)\, f\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}\cdots\right)\right] = E_{\alpha,0}[V_n f(V_1, V_2, \cdots)] \qquad (116)$$

for all $n = 1, 2\cdots$ and all non-negative product measurable functions f. Here E denotes expectation relative to P governing the stable(α) subordinator (τ_s), and $E_{\alpha,0}$ denotes expectation relative to $P_{\alpha,0}$ governing (V_n) with $\mathrm{PD}(\alpha,0)$ distribution. In particular, from Proposition 34 for $0<\alpha<1$ and $\theta = 0$ we obtain for all $t > 0$

$$P(N_t = n) = \int_0^\infty dz\, e^{-z}\phi_\alpha(z)^{n-1}\psi_\alpha(z)^{-n} \qquad (117)$$

This is a companion of a result of Scheffer [53], which can be expressed in present notation as follows:

$$P(N_{D_t} = n) = \alpha \int_0^\infty dz\, z^{-1}(1 - e^{-z})\phi_\alpha(z)^{n-1}\psi_\alpha(z)^{-n} \tag{118}$$

Here $N_t - 1$ is the number of excursions completed by time t whose lengths exceed $t - G_t$, while $N_{D_t} - 1$ is the smaller number of such excursions whose lengths exceed the length $D_t - G_t$ of the excursion straddling time t, for G_t and D_t defined in (68). Formula (118) is a consequence of the following analog of (116), established in [51],

$$E\left[1(N_{D_t} = n)\, f\left(\frac{V_1(D_t)}{D_t}, \frac{V_2(D_t)}{D_t} \cdots\right)\right] = E_{\alpha,0}[-\alpha \log(1 - V_n) f(V_1, V_2, \cdots)] \tag{119}$$

which for $f = 1$ gives

$$P(N_{D_t} = n) = E_{\alpha,0}[-\alpha \log(1 - V_n)] = \alpha \sum_{p=1}^{\infty} \frac{1}{p} E_{\alpha,0}[V_n^p] \tag{120}$$

Evaluating $E_{\alpha,0}[V_n^p]$ using (48) now yields (118). Using (119) and (49) we obtain the following asymptotic formulae as $n \to \infty$:

$$P(N_{D_t} = n) \sim \alpha P(N_t = n) \sim \frac{\alpha ,(\frac{1}{\alpha} + 1)}{,(1 - \alpha)^{1/\alpha}} \frac{1}{n^{1/\alpha}} \tag{121}$$

where $a(n) \sim b(n)$ means $a(n)/b(n) \to 1$ as $n \to \infty$. To illustrate, in the Brownian case ($\alpha = \frac{1}{2}$) the numerical values in the following table were obtained using a four line *Mathematica* program which evaluated the integrals (117) and (118) numerically after definition of ϕ_α and ψ_α in terms of *Mathematica's* incomplete gamma function. The numerical values for N_{D_t} agree with those of Scheffer [53]. The asymptotic formulae as $n \to \infty$ are read from (121). For $n = 4$ the asymptotic formula gives the approximations 0.0398 and 0.0199, which are already very close to the values of $P(N_t = 4)$ and $P(N_{D_t} = 4)$ shown in the table.

Distribution of N_t and N_{D_t} for Brownian motion

n	1	2	3	4	$\cdots$	$\to \infty$
$P(N_t = n)$	0.6265	0.1430	0.0630	0.0356	$\cdots$	$\sim 2/(\pi n^2)$
$P(N_{D_t} = n)$	0.8003	0.0812	0.0334	0.0185	$\cdots$	$\sim 1/(\pi n^2)$

A simplified approach to (117) and (118), which gives a probabilistic interpretation of the integrals in these formulae, can be made as follows. Let T be an exponential variable with rate 1 independent of the subordinator (τ_s). It is clear by scaling that N_t for each t has the same distribution as N_T, so it is enough to establish the formulae with T instead of t. By consideration of a Poisson process of marked excursions as in Section 3 of [50], it is found that $T - G_T$ has gamma $(1-\alpha)$ distribution, and given $T - G_T = z$ that N_T has geometric distribution with parameter $p_\alpha(z)$ as in (114). That is to say,

$$P(T - G_T \in dz, N_T = n) = \frac{1}{\Gamma(1-\alpha)} z^{-\alpha} e^{-z} dz\, (1 - p_\alpha(z))^{n-1} p_\alpha(z) \quad (122)$$

which gives a natural disintegration of (117) with t replaced by T. A similar argument with $D_T - G_T$ instead of $T - G_T$ yields

$$P(D_T - G_T \in dz, N_{D_T} = n) = \frac{\alpha}{\Gamma(1-\alpha)} z^{-\alpha} (1 - e^{-z})\, dz (1 - p_\alpha(z))^{n-1} p_\alpha(z) \quad (123)$$

which is the corresponding disintegration of (118). To summarize, the distributions of N_t and N_{D_t} are two different integral mixtures of geometric(p) distributions on $\{1, 2, \cdots\}$; the mixing distribution is that of $p_\alpha(T - G_T)$ in the case of N_t, and that of $p_\alpha(D_T - G_T)$ in the case of N_{D_t}.

6.4 Interpretation in the bridge case $\theta = \alpha$

In the case $\theta = \alpha$, corresponding to a Brownian or Bessel bridge, the distribution of N described in (113) can be understood as follows. Starting with a $(2 - 2\alpha)$-dimensional Bessel bridge of length 1, whose ranked excursion lengths are $V_1 > V_2 > \cdots$, let U be uniform on $[0, 1]$ independent of the bridge, and let $V_N = D_U - G_U$ be the length of the excursion interval (G_U, D_U) that contains time U. So V_N is a length-biased pick from the sequence of lengths (V_n). Then, as shown in Aldous-Pitman [1] for $\alpha = \frac{1}{2}$, and in [45] for $0 < \alpha < 1$, the joint distribution of $(G_U, D_U - G_U, 1 - D_U)$ is Dirichlet with parameters $(\alpha, 1 - \alpha, \alpha)$, and conditionally given $(G_U, D_U - G_U, 1 - D_U)$ the process B decomposes into three independent components: two bridges of lengths G_U and $1 - D_U$, and an excursion of length $D_U - G_U$. Let $V_1' > V_2' > \cdots$ denote the ranked excursion lengths up to time G_U, and let $V_1'' > V_2'' > \cdots$ denote the ranked excursion lengths derived from the interval

$(D_U, 1)$. Note that the sequence $V_1 > V_2 > \cdots$ is obtained by ranking the set of lengths $V_1', V_2', \cdots, V_N, V_1'', V_2'', \cdots$, and that

$$N - 1 = N' + N''$$

where N' is the number of i such that $V_i' > V_N$, and N'' is the number of i such that $V_i'' > V_N$. Now, if we introduce a gamma$(1+\alpha)$ random variable $Z_{1+\alpha}$ independent of the bridge, then $Z_{1+\alpha}G_U, Z_{1+\alpha}V_N$ and $Z_{1+\alpha}(1-D_U)$ are three independent gamma variables with parameters $\alpha, 1-\alpha$ and α respectively, and the three random components $Z_{1+\alpha}(V_1', V_2', \cdots)$, $Z_{1+\alpha}(V_1'', V_2'', \cdots)$ and $Z_{1+\alpha}V_N$ are mutually independent. Moreover, the two infinite sequences are identically distributed, and the joint law of either of these sequences with $Z_{1+\alpha}V_N$ is identical to the joint law of $(V_1(G_T), V_2(G_T), \cdots)$ with $T - G_T$ as considered in the previous section for an unconditioned Bessel process and an independent standard exponential variable T. It is now follows from the previous discussion that the formula $N - 1 = N' + N''$ presents $N-1$ as the sum of two random variables which given $Z_{1+\alpha}V_N = z$ are i.i.d. geometric with parameter $p_\alpha(z)$. Thus we recover the result (113) in the bridge case $\theta = \alpha$: the distribution of $N-1$ is an integral mixture of negative binomial distributions with shape parameter 2.

7 The Markov chain derived from $\text{PD}(\alpha, \theta)$

Starting from any ranked sequence of random variables $V_1 \geq V_2 \cdots$ with $\sum_n V_n = 1$, define new variables R_n and Y_n as in (20) and (44). Note the relations (22) and (45) which allow any one of the sequences $(V_n), (Y_n)$ and (R_n) to be recovered from any of the others. Note also the relations

$$Y_n = (1 + R_n + R_n R_{n+1} + \cdots)^{-1} = \frac{Y_{n+1}}{Y_{n+1} + R_n}; \quad R_n = \frac{Y_{n+1}(1 - Y_n)}{Y_n} \quad (124)$$

and the a priori constraints

$$0 \leq R_n \leq 1, \quad 1 + R_1 + R_1 R_2 + \cdots < \infty, \quad 0 \leq Y_{n+1} \leq Y_n/(1 - Y_n) \quad (125)$$

7.1 The cases of $\text{PD}(\alpha, 0)$ and $\text{PD}(0, \theta)$

The following proposition is suggested by results of Vervaat [60, 61] and Vershik [57]:

Proposition 35 *Suppose that $R_1, R_2, \cdots$ are independent, and satisfy* (125) *a.s.. Then (Y_n) is a Markov chain, typically with inhomogeneous transition probabilities. If the R_n are identically distributed, then (Y_n) is stationary, with homogeneous transition probabilities. If R_n has density*

$$P(R_n \in dr) = f_n(r)dr, \tag{126}$$

then (Y_n) has co-transition probabilities

$$\frac{P(Y_n \in dy_n | Y_{n+1} = y_{n+1})}{dy_n} = 1\left(0 < y_{n+1} < \frac{y_n}{\bar{y}_n}\right) f_n\left(\frac{y_{n+1}\bar{y}_n}{y_n}\right)\frac{y_{n+1}}{y_n^2} \tag{127}$$

where $\bar{y}_n = 1 - y_n$

Proof. Since from (124) Y_{n+k} is a function of $R_{n+1}, R_{n+2}, \ldots$, it is immediate that $Y_n = Y_{n+1}/(Y_{n+1} + R_n)$ is conditionally independent of $Y_{n+1}, Y_{n+2}, \ldots$ given Y_{n+1}. This yields the Markov property in reverse time. The formula for the co-transition probabilities is immediate by change of variable. Clearly, (Y_n) is stationary if (R_n) is i.i.d.. □

Recall that $P_{\alpha,\theta}$ governs (V_n) according to the PD(α, θ) distribution. According to Theorem 7, under $P_{\alpha,0}$ for $0 < \alpha < 1$, the R_n are independent with beta$(n\alpha, 1)$ distributions. Thus Proposition 35 implies that under $P_{\alpha,0}$ the sequence (Y_n) is Markov with inhomogeneous co-transition probabilities that may be read from the Proposition. The transition probabilities in the forwards direction can then be written down using Bayes' rule, in terms of the density functions $p_{\alpha,0,n}(u)$, where for general (α, θ) we define

$$p_{\alpha,\theta,n}(u) = P_{\alpha,\theta}(V_n \in du)/du. \tag{128}$$

These densities are fairly complicated however. See Section 8.1.

This result under PD$(\alpha, 0)$ for $0 < \alpha < 1$ is analogous to the following result of Vershik and Shmidt [59], Ignatov [32]: under PD$(0, \theta)$ for $\theta > 0$, the sequence (Y_n) is Markov with homogeneous transition probabilities

$$\frac{P_{0,\theta}(Y_{n+1} \in dy | Y_n = x)}{dy} = 1\left(0 < y < \frac{x}{\bar{x}} \wedge 1\right)\theta x^{-1}\bar{x}^{\theta-1}\frac{p_{0,\theta,1}(y)}{p_{0,\theta,1}(x)} \tag{129}$$

While in the PD$(0, \theta)$ case the transition probabilities of the chain (Y_n) are homogeneous, the chain is not stationary. According to [59, 32], the stationary probability density for this chain is given by

$$p^*_{0,\theta}(x) = K^{-1}_{0,\theta}x^{-\theta}p_{0,\theta,1}(x) \tag{130}$$

where $K_{0,\theta}$ is a normalization constant. As shown by Ignatov [32], results of Vervaat [60] and Watterson [63] imply that

$$K_{0,\theta} = \, , \, (\theta + 1)e^{\theta\gamma} \tag{131}$$

where $\gamma = -, '(1) = 0.5771 \cdots$ is Euler's constant, and that

$$p^*_{0,\theta}(x) = P^*_{0,\theta}(V_1 \in dx)/dx \tag{132}$$

where $P^*_{0,\theta}$ makes (R_n) a sequence of i.i.d. beta$(\theta, 1)$ random variables, and

$$V_1 = (1 + R_1 + R_1R_2 + \cdots + R_1R_2R_3 + \cdots)^{-1}. \tag{133}$$

The densities $p_{0,\theta,1}(x)$ and $p^*_{0,\theta}(x)$ are then determined by the $P^*_{0,\theta}$ distribution of $\Sigma_1 := (1 - V_1)/V_1$, which is the infinitely divisible law with Laplace transform

$$E^*_{0,\theta}[\exp(-\lambda\Sigma_1)] = \exp\left(-\theta \int_0^1 dx \frac{(1 - e^{-\lambda x})}{x}\right) \tag{134}$$

Most of these results were obtained earlier in the special case $\theta = 1$, which arises in applications to combinatorics and number theory (see Dickman [13], Shepp-Lloyd [54], Goncharov [22], Billingsley [6], Vershik and Shmidt [58, 59, 57],Donnelly-Grimmett [15]).

It is easily verified using Proposition 35 that $P^*_{0,\theta}$ makes (Y_n) a stationary Markov chain with the same homogeneous transition probabilities as those displayed in (129) under $P_{0,\theta}$. Consequently, the above results are largely summarized by the following identity: for all positive product measurable functions f,

$$E_{0,\theta}[\, f(Y_1, Y_2, \cdots)\,] = K_{0,\theta}E^*_{0,\theta}[Y_1^\theta f(Y_1, Y_2, \cdots)] \tag{135}$$

Note that since $V_1 = Y_1$ and the (V_n) sequence can be recovered from the (Y_n) sequence and vice-versa, formula (135) holds just as well with Y_n replaced everywhere by V_n. The same is true of formula (136) below.

7.2 Extension to PD(α, θ)

The following theorem, which is an amplification of Theorem 14, generalizes the entire collection of results described in the previous subsection to the full two-parameter family PD(α, θ) :

Theorem 36 *Let sequences* (V_n), (R_n) *and* (Y_n) *be related by* (20), (44), (124). *For* $0 \le \alpha < 1, \theta > -\alpha$, *let* $P_{\alpha,\theta}$ *govern* (V_n) *with* PD(α,θ) *distribution, and let* $P^*_{\alpha,\theta}$ *govern* $(R_1, R_2, \cdots)$ *as a sequence of independent random variables, such that* R_n *has beta*$(\theta + n\alpha, 1)$ *distribution. Then*

(i) *for every product measurable function* f,

$$E_{\alpha,\theta}[\, f(Y_1, Y_2, \cdots)\,] = K_{\alpha,\theta} E^*_{\alpha,\theta}[Y_1^\theta f(Y_1, Y_2, \cdots)] \tag{136}$$

where $K_{0,\theta}$ *is given in* (131) *and*

$$K_{\alpha,\theta} = , (\theta+1), (1-\alpha)^{\theta/\alpha} \qquad (0 < \alpha < 1, \theta > -\alpha) \tag{137}$$

(ii) *Both* $P = P_{\alpha,\theta}$ *and* $P = P^*_{\alpha,\theta}$ *govern* (Y_n) *as a Markov chain with the same forwards transition probabilities, given by* (129) *for* $\alpha = 0$ *and as follows for* $0 < \alpha < 1$:

$$\frac{P(Y_{n+1} \in dy_{n+1} | Y_n = y_n)}{dy_{n+1}} = y_n^{-\alpha-1}(1-y_n)^{n\alpha+\theta-1} \frac{r(\alpha, \theta + n\alpha, y_{n+1})}{r(\alpha, \theta + n\alpha - \alpha, y_n)} \tag{138}$$

for $0 < y_n < 1$, $0 < y_{n+1} < y_n/(1-y_n)$, *and* 0 *otherwise, where*

$$r(\alpha,\theta,y)dy = , (\frac{\theta}{\alpha} + 1) y^\theta P^*_{\alpha,\theta}(V_1 \in dy) = C^{-1}_{\alpha,\theta} P_{\alpha,\theta}(V_1 \in dy) \tag{139}$$

for $C_{\alpha,\theta}$ *as in* (42), *and* $V_1 = Y_1$.

(iii) *The* $P^*_{\alpha,\theta}$ *distribution of* $\Sigma_1 := (1 - V_1)/V_1$ *is infinitely divisible, with Laplace transform given for* $\alpha = 0, \theta > 0$ *by* (134), *and for* $0 < \alpha < 1, \theta > -\alpha$ *by*

$$E^*_{\alpha,\theta}[\exp(-\lambda \Sigma_1)] = \left(\frac{1}{\psi_\alpha(\lambda)} \right)^{\frac{\theta}{\alpha}+1} \tag{140}$$

for ψ_α *as in* (33).

Remark 37 For $0 < \alpha < 1$, the function $r(\alpha, \theta, y)$ is determined by the first equality in (139) and the Laplace transform (140). The last expression in (139) and Proposition 45 in the next section yield alternative formulae for $r(\alpha,\theta,y)$. For $\alpha = 0$, the chain (Y_n) is stationary and homogeneous under $P^*_{0,\theta}$, whereas in case $0 < \alpha < 1$ the chain is non-homogeneous, and the distribution of Y_n depends on n. See Section 7.3 below regarding the asymptotic distribution of Y_n as $n \to \infty$.

Remark 38 As the results for $\alpha = 0$ are known, we shall assume for the proof that $0 < \alpha < 1$. We note however that the results for $\alpha = 0$ can be recovered by passage to the limit as $\alpha \downarrow 0$ for fixed θ, using (105).

Proof of Theorem 36. Let $0 < \alpha < 1$.

(i). From the basic absolute continuity relation (41), for all measurable $f \geq 0$

$$E_{\alpha,\theta}[f(Y_1, Y_2, \cdots)] = C_{\alpha,\theta} E_{\alpha,0}[\, L^{\theta/\alpha} f(Y_1, Y_2, \cdots)\,] \tag{141}$$

where L is the local time variable, which can be expressed from (23) as

$$L = Y_1^{\alpha} \lim_{n\to\infty} n(R_1 \cdots R_n)^{\alpha} \qquad (P_{\alpha,\theta} \text{ a.s. for all } \theta > -\alpha) \tag{142}$$

On the other hand, since both $P^*_{\alpha,\theta}$ and $P_{\alpha,0}$ make $R_1, \cdots, R_n$ a sequence of independent beta variables, calculating the ratio of the two product densities gives

$$E^*_{\alpha,\theta}[f(R_1, \cdots, R_n)] = \frac{\Gamma(\frac{\theta}{\alpha} + n + 1)}{\Gamma(\frac{\theta}{\alpha} + 1)\Gamma(n+1)} E_{\alpha,0}[(R_1 \cdots R_n)^{\theta} f(R_1, \cdots, R_n)]. \tag{143}$$

Passage to the limit as $n \to \infty$, using $\Gamma(\theta/\alpha + n + 1)/\Gamma(n+1) \sim n^{\theta/\alpha}$, martingale convergence, and (142), yield

$$E^*_{\alpha,\theta}[f(R_1, R_2, \cdots)] = \Gamma(\tfrac{\theta}{\alpha} + 1)^{-1} E_{\alpha,0}[\, L^{\theta/\alpha} Y_1^{-\theta} f(R_1, R_2, \cdots)] \tag{144}$$

a formula which holds just as well with $f(Y_1, Y_2, \cdots)$ instead of $f(R_1, R_2, \cdots)$, due to (124). Comparison of (141) and (144) yields (136).

(ii). According to Proposition 35, (Y_n) is a Markov chain under $P^*_{\alpha,\theta}$ with transition probabilities that can be written down using the general form of the co-transition probabilities (127), and the prescribed beta density of R_n which is $f_n(x) = (\theta + n\alpha)x^{\theta + n\alpha - 1}$ for $0 < x < 1$. Bayes' rule then yields the forwards transition probabilities of the form (138), for $r(\alpha, \theta, y)$ defined by the first equality in (139), after using the formula

$$P^*_{\alpha,\theta}(Y_n \in dy) = P^*_{\alpha,\theta+(n-1)\alpha}(Y_1 \in dy). \tag{145}$$

This follows from (124), since by definition the $P^*_{\alpha,\theta}$ distribution of $R_n, R_{n+1}, \cdots$ is the $P^*_{\alpha,\theta+(n-1)\alpha}$ distribution of $R_1, R_2, \cdots$. The second equality in (139) for $r(\alpha, \theta, y)$ is immediate from (136) and the formula (29) for $C_{\alpha,\theta}$ in (41). Since the density factor $dP^*_{\alpha,\theta}/dP_{\alpha,\theta} = K_{\alpha,\theta}Y_1^\theta$ is a function of Y_1, it is clear without further calculation that (Y_n) must be Markov under $P_{\alpha,\theta}$ with the same transition probabilities as under $P^*_{\alpha,\theta}$.

(iii). To obtain the formula (140) for the Laplace transform of $\Sigma_1 := (1-V_1)/V_1$ use (144) to compute

$$E^*_{\alpha,\theta}[\exp(-\lambda\Sigma_1)] = , (\tfrac{\theta}{\alpha}+1)^{-1}E_{\alpha,0}[\, X_1^{\theta/\alpha}\exp(-\lambda\Sigma_1)\,] \qquad (146)$$

where $X_1 = LV_1^{-\alpha}$ has exponential distribution with rate 1. But from (67)

$$E_{\alpha,0}[\exp(-\lambda\Sigma_1)|X_1] = \exp[-X_1(\psi_\alpha(\lambda)-1)]$$

and using this expression in (146) yields (140). □

Immediately from the above theorem, we derive the formula of the following corollary, which extends formulae of Vershik and Shmidt [59], and Ignatov [32] in the case $\alpha = 0$. The Markov property of (Y_n) under $P_{\alpha,\theta}$ is evident by inspection of this formula. This formula can also be derived by suitable changes of variables and integration from Proposition 45, after changing variables and integrating out t. Combined with Proposition 35, this gives an alternative approach to the previous theorem.

Corollary 39 *The $P_{\alpha,\theta}$ joint density of $Y_1,\cdots,Y_n$ is given by the formula*

$$P_{\alpha,\theta}(Y_1\in dy_1,\cdots,Y_n\in dy_n)/\Pi_{i=1}^n dy_i =$$

$$C_{\alpha,\theta}\alpha^{n-1}\Pi_{i=1}^{n-1}[y_i^{-\alpha-1}(1-y_i)^{i\alpha+\theta-1}1(y_{i+1}<y_i/(1-y_i))]\, r(\alpha, n\alpha-\alpha+\theta, y_n)$$

for $r(\alpha,\theta,y)$ defined by (139)*.*

Remark 40 Since $P_{\alpha,0} = P^*_{\alpha,0}$ for all $0<\alpha<1$, the special case $0<\alpha<1, \theta=0$ of formula (145) allows computation of the $P_{\alpha,0}$ distribution of Y_n:

$$P_{\alpha,0}(Y_n\in dy) = \frac{1}{(n-1)!}y^{-(n-1)\alpha}\, r(\alpha, n\alpha-\alpha, y)\, dy \qquad (147)$$

This result can also be read from formula (80). In particular, the moments of Y_n derived from PD$(\alpha,0)$ are given by the expression

$$E_{\alpha,0}Y_n^p = \frac{1}{(n-1)!}E_{\alpha,0}[V_1^{p-(n-1)\alpha}L^{n-1}] \qquad (148)$$

which can be evaluated using (84).

Remark 41 Note that if $(\tilde{Y}_n)$ are the independent factors as in (4) derived from the size biased permutation $(\tilde{V}_n)$ of a $\text{PD}(\alpha, \theta)$ sequence (V_n), then for each $k = 1, 2 \cdots$ the sequence $(\tilde{Y}_{n+k}, n = 1, 2 \cdots)$ has the same distribution as the independent factors derived similarly from the size biased presentation of $\text{PD}(\alpha, \theta + k\alpha)$. On the other hand, the sequence $(Y_{n+k}, n = 1, 2 \cdots)$ is Markovian with the same sequence of inhomogeneous transition probabilities as (Y_n) derived from a $\text{PD}(\alpha, \theta + k\alpha)$, but the initial distribution is different. This distinction appears already for $\alpha = 0$: then (Y_n) has stationary transition probabilities, but the distribution of Y_n varies with n, only approaching the stationary distribution in the limit as $n \to \infty$.

To illustrate by a concrete example, $(Y_2, Y_3, \cdots)$ derived from excursions of an unconditioned Bessel process is a Markov chain with exactly the same inhomogeneous transition function as $(Y_1, Y_2, \cdots)$ derived from the corresponding bridge. However Y_2 for the unconditioned process does not have the same law as Y_1 for the bridge.

7.3 Asymptotic behaviour of the $\text{PD}(\alpha, \theta)$ chain

It was shown by Vershik and Shmidt [59] for $\theta = 1$ and Ignatov [32] for general $\theta > 0$ that the $P_{0,\theta}$ distribution of Y_n converges to the stationary distribution (130) of the Markov chain. For $0 < \alpha < 1, \theta > -\alpha$, the asymptotic behaviour of the distribution of Y_n can be derived as follows from the relation $Y_n = 1/(1 + \Sigma_n)$ and the description of the $P_{\alpha,0}$ distribution of Σ_n provided by Proposition 10(ii). According to that proposition, under $P_{\alpha,0}$ the random variable Σ_n is the sum of n independent copies of Σ_1, which has finite moments of all orders, obtained by differentiation of its Laplace transform (36). In particular

$$E_{\alpha,0}(\Sigma_1) = \frac{\alpha}{1-\alpha}$$

and a strong law of large numbers implies that

$$\frac{\Sigma_n}{n} \to \frac{\alpha}{1-\alpha} \quad P_{\alpha,0} \text{ a.s.}$$

hence also $P_{\alpha,\theta}$ a.s. for all $\theta > -\alpha$ by Proposition 13. Similarly, the central limit theorem implies that the $P_{\alpha,0}$ distribution of

$$\sqrt{n}\left(\frac{\Sigma_n}{n} - \frac{\alpha}{1-\alpha}\right)$$

converges to normal with mean zero and variance

$$Var_{\alpha,0}(\Sigma_1) = \frac{\alpha}{(2-\alpha)(1-\alpha)^2}$$

A standard argument shows that this limit law under $P_{\alpha,0}$ is mixing in the sense of [2]. That is to say the same limit distribution is obtained after a change of measure to any distribution Q that is absolutely continuous with respect to $P_{\alpha,0}$, in particular, for $Q = P_{\alpha,\theta}$ for all $\theta > -\alpha$. Translating these results in terms of $Y_n = 1/(1+\Sigma_n)$ yields the following proposition:

Proposition 42 *Under $P_{\alpha,\theta}$ for all $0 < \alpha < 1$ and $\theta > -\alpha$,*

$$nY_n \to \frac{1-\alpha}{\alpha} \qquad a.s. \tag{149}$$

and the distribution of

$$\sqrt{n}\left(nY_n - \frac{1-\alpha}{\alpha}\right) \tag{150}$$

converges to the normal distribution with mean zero and variance $\alpha^{-2}(2-\alpha)^{-2}$.

These asymptotics for Y_n may be compared with the corresponding behaviour of the independent factors $(\tilde{Y}_n)$ as in (4). From the beta $(1-\alpha, \theta+n\alpha)$ distribution of $\tilde{Y}_n$ under $P_{\alpha,\theta}$ one gets:

$$E_{\alpha,\theta}(\tilde{Y}_n) = \frac{1-\alpha}{1+\theta+(n-1)\alpha}$$

For $0 < \alpha < 1, \theta > -\alpha$, this makes

$$E_{\alpha,\theta}(n\tilde{Y}_n) \to \frac{1-\alpha}{\alpha} \text{ as } n \to \infty$$

More precisely, the asymptotic distribution of $\alpha n \tilde{Y}_n$ is gamma $(1-\alpha)$ So Y_n and $\tilde{Y}_n$ are both of order $1/n$ for large n, their means are asymptotically the same, but their asymptotic distribution is different.

8 Some Results for a General Subordinator

We collect in this section some results regarding interval lengths $V_n(t)$ derived for a general subordinator (τ_s) as in Section 1.2, which in the stable and gamma cases are related to PD(α, θ) .

8.1 Perman's Formula

Let $\Delta_1 \geq \Delta_2 \geq \cdots$ be the ranked jumps up to time 1 of a drift-free subordinator $(\tau_s, s \geq 0)$. Put $V_n = \Delta_n/\tau_1$. Perman [43] found a formula for the $(n+1)$-dimensional joint density

$$p_n(t, v_1, \cdots, v_n) = P(\tau_1 \in dt, V_1 \in dv_1, \cdots, V_n \in dv_n)/dt\, dv_1 \cdots dv_n \quad (151)$$

assuming the Lévy measure Λ of (τ_s) has a density h with respect to Lebesgue measure on $(0, \infty)$. Perman's formula is as follows. For $n \geq 2$

$$p_n(t, v_1, v_2, \cdots, v_n) = \frac{t^{n-1} h(tv_1) h(tv_2) \cdots h(tv_{n-1})}{\tilde{v}_n} p_1 \left(t\tilde{v}_n, \frac{v_n}{\tilde{v}_n} \right) \quad (152)$$

for $t > 0$ and $0 < v_1 < v_2 < \cdots < v_n < 1$, $\sum_i v_i < 1$, where

$$\tilde{v}_n = 1 - v_1 - v_2 - \cdots - v_{n-1},$$

and

$$p_1(t, v) = P(\tau_1 \in dt, V_1 \in dv)/dtdv \quad (153)$$

is the unique solution of the integral equation

$$p_1(t, v) = th(tv) \int_0^{\frac{v}{1-v} \wedge 1} p_1(t(1-v), u) du \quad (154)$$

for $t > 0$ and $v \in (0, 1)$.

Proposition 43 *Let $f(t) := P(\tau_1 \in dt)/dt$ denote the density of τ_1, and define a sequence of non-negative functions $f_n(t, u), t > 0, 0 < u < 1$ inductively as follows:*

$$f_1(t, u) = th(tu) f(t\bar{u}) \quad (155)$$

where $\bar{u} = 1 - u$, and for $n = 1, 2, \cdots$

$$f_{n+1}(t, u) = 1(u \leq 1/n) th(tu) \int_{u/\bar{u}}^1 dv\, f_n(t\bar{u}, v) \quad (156)$$

The joint density $p_1(t, v)$ appearing in (153) and (152) is given by the formula

$$p_1(t, v) = \sum_1^\infty (-1)^{n+1} f_n(t, v) \quad (157)$$

where all but the first n terms of the sum are zero if $v > 1/(n+1)$.

Proof. This is straightforward by induction on n, using Perman's integral equation (154).

Remark 44 Integrating formula (157) from u to 1 gives a series expression for $P(V_1 > u, \tau_1 \in dt)$. It can be shown by induction that this series is identical to that obtained by Perman by a different method in formula (8) of [43].

Suppose for the rest of this section that (τ_s) is a stable subordinator of index α, as in (11). Then the density $h(x)$ of the Lévy measure is

$$h(x) = \alpha C x^{-\alpha-1}, \qquad (x > 0) \tag{158}$$

and from (29) the density $f_\alpha(t)$ of τ_1 is characterized by its negative moments via the following formula: for all real $theta > -\alpha$

$$\int_0^\infty t^{-\theta} f_\alpha(t)dt = E(\tau_1^{-\theta}) = \frac{1}{C^{\theta/\alpha}C_{\alpha,\theta}} = \frac{,(\frac{\theta}{\alpha}+1)}{,(\theta+1)} \frac{1}{(C,(1-\alpha))^{\theta/\alpha}} \tag{159}$$

Proposition 45 *Let (V_n) have* PD$(\alpha, 0)$ *distribution, and let Σ be defined as in (24), so Σ is the sum of the points Δ_n of the* PRM *Λ_α derived from (V_n). Then the joint density of $(\Sigma, V_1, \cdots, V_n)$ is the function $p_n(t, v_1, v_2, \cdots, v_n)$ given by Perman's formula (152) with $h(x)$ defined by (158) and $p_1(t, v)$ derived as in Proposition 43 from $f(x) = f_\alpha(x)$ defined by (159). For (V_n) with* PD(α, θ) *distribution, for $0 < \alpha < 1$, $\theta > -\alpha$, the corresponding joint density is $c_{\alpha,\theta}\, t^{-\theta} p_n(t, v_1, v_2, \cdots, v_n)$ where $c_{\alpha,\theta} = C^{\theta/\alpha}C_{\alpha,\theta}$. .*

Proof. This is an immediate consequence of Propositions 9, 43 and 13. □

Integrating out t in the above $(n+1)$-dimensional joint density gives an expression for the n-dimensional joint density of $(V_1, \cdots, V_n)$ for a PD(α, θ) distributed sequence (V_n). In particular, for $n = 1$ we obtain Proposition 19 as follows:

Proof of Proposition 19. Proposition 45 combined with Proposition 43 yields formula (52) with the nth term of the sum replaced by the expression $(-1)^{n+1} c_{\alpha,\theta} \int_0^\infty t^{-\theta} f_{n,\alpha}(t, u)dt$ where $f_{n,\alpha}(t, u)$ is the $f_n(t, u)$ defined inductively by Proposition 43 starting from $f(t) = f_\alpha(t)$ as in (159). Chasing these definitions yields the expression (53) by making a suitable change of variable to simplify the integral with respect to t using (159). □

8.2 Laplace transforms for some infinite products

Let $V_n(T)$ be derived as in (6) from the closed range Z of a subordinator (τ_s) with Lévy measure Λ as in (7). The formulae of the following proposition serve to characterize the laws of the sequences $(V_n(s))$ and $(V_n(\tau_t)/\tau_t)$ for all $s > 0$ and $t > 0$. A formula like (160) involving just $V_1(s)$ appears as Theorem 2.1 of Knight [35]. See also formula (76) of Kingman [33] for an expression similar to (162) related to $V_1(\tau_t)/\tau_t$.

Proposition 46 *For each measurable function $g : (0,\infty) \to [0,1]$ such that $\int_0^\infty \Lambda(dv)(1-g(v)) < \infty$, and $\lambda \geq 0$,*

$$\int_0^\infty ds\, e^{-\lambda s} E\left[\prod_n g(V_n(s))\right] = \frac{\int_0^\infty du e^{-\lambda u}\Lambda(u,\infty)g(u)}{\int_0^\infty \Lambda(dv)(1-e^{-\lambda v}g(v))} \tag{160}$$

$$\int_0^\infty ds\, e^{-\lambda s} E\left[\prod_n g\left(\frac{sV_n(\tau_t)}{\tau_t}\right)\right] = \tag{161}$$

$$\int_0^\infty du \left(t\int_0^\infty \Lambda(dv)e^{-\lambda uv}g(uv)v\right)\exp\left(-t\int_0^\infty \Lambda(dw)(1-e^{-\lambda uw}g(uw)\right) \tag{162}$$

Proof. By considering these identities with $e^{-\lambda s}g(s)$ instead of $g(s)$ it is enough to prove them for $\lambda = 0$. The left side of (160) then equals

$$E\left[\sum_{u>0}\int_{\tau_{u-}}^{\tau_u} ds \left(\prod_m g(V_m(\tau_{u-})\right) g(s-\tau_{u-})\right]$$

which, using the basic compensation formula of excursion theory, equals

$$E\left[\int_0^\infty du\,\left(\prod_m g(V_m(\tau_{u-}))\right)\right]\left(\int_0^\infty dv\Lambda(v,\infty)g(v)\right)$$

Now (160) follows easily after evaluating the expectation above using Fubini's theorem and the formula

$$E\left[\prod_n g(V_n(\tau_{u-}))\right] = E\left[\prod_n g(V_n(\tau_u))\right] = \exp\left(-u\int_0^\infty \Lambda(dx)(1-g(x))\right) \tag{163}$$

which is an expression of the fact that the $V_n(\tau_u)$ are the points of a PRM $(u\Lambda)$ (Kingman [34], (3.35)). Turning to (161), the change of variables $s = u\tau_t$ allows (161) for $\lambda = 0$ to be rewritten as

$$\int_0^\infty du\, E\left[\tau_t \prod_n g(uV_n(\tau_t))\right]$$

The integrand can be evaluated using (163) with t instead of u and $g(ux)e^{-\lambda x}$ instead of $g(x)$, by differentiation with respect to λ at $\lambda = 0$. The result is (162). □

For a stable (α) subordinator with $\Lambda = \Lambda_\alpha$ as in (11), it is easily verified that the expression in (162) equals the right side of the expression in (160), which proves the identity in law of the two sequences featured in Proposition 5. Note also that (163) and hence (160) can be verified also for measurable $g : (0,\infty) \to [0,\infty)$ such that $0 < \int_0^\infty \Lambda(dv)(1 - g(v)) < \infty$ provided the integral is absolutely convergent. Thus we obtain the following Corollary regarding the expectation of an infinite product derived from (V_n) with PD$(\alpha, 0)$ distribution:

Corollary 47 *For $0 < \alpha < 1$ and $g : (0,\infty) \to [0,\infty)$ such that*

$$0 < \int_0^\infty \frac{dv}{v^{\alpha+1}}(1 - g(v)) < \infty \tag{164}$$

and the integral is absolutely convergent, define

$$K_g(\alpha, \lambda) = \int_0^\infty \frac{dv}{v^{\alpha+1}}(1 - e^{-\lambda v} g(v)) \tag{165}$$

$$K_g'(\alpha, \lambda) = \frac{d}{d\lambda} K_g(\alpha, \lambda) = \int_0^\infty \frac{dv}{v^\alpha} e^{-\lambda v} g(v) \tag{166}$$

Then

$$\int_0^\infty ds\, e^{-\lambda s} E_{\alpha,0}\left[\prod_n g(sV_n)\right] = \frac{K_g'(\alpha,\lambda)}{\alpha K_g(\alpha,\lambda)} \tag{167}$$

To illustrate, taking $g(x) = \exp(-\kappa x^p)$ for $\kappa > 0$ and $p > 1$ gives a double Laplace transform which determines the distribution of $\sum_n V_n^p$ for a

PD$(\alpha, 0)$ distributed (V_n). Unfortunately, such transforms seem difficult to invert. For g a polynomial with non-negative coefficients, say

$$g(x) = 1 + \sum_{j=1}^{k} a_j x^j$$

we find that

$$K_g(\alpha, \lambda) = \frac{,(1-\alpha)}{\alpha}\lambda^{\alpha} - \sum_{j=1}^{k} a_j ,(j-\alpha)\lambda^{\alpha-j}$$

and hence that the Laplace transform in (167) is

$$\frac{K_g'(\alpha, \lambda)}{\alpha K_g(\alpha, \lambda)} = \frac{1}{\lambda}\left(1 + \frac{\sum_{j=1}^{k} j ,(j-\alpha) a_j \lambda^{k-j}}{,(1-\alpha)\lambda^k - \alpha \sum_{j=1}^{k} ,(j-\alpha) a_j \lambda^{k-j}}\right) \tag{168}$$

In particular cases this transform can be inverted to obtain for example

$$E_{\alpha,0}\left[\prod_n (1 + aV_n^p)\right] = 1 + \frac{p}{\alpha}\sum_{k=1}^{\infty}\frac{1}{(pk)!}\left(\frac{\alpha ,(p-\alpha)}{,(1-\alpha)}\right)^k a^k \tag{169}$$

which for $p = 1$ and $p = 2$ becomes

$$E_{\alpha,0}\left[\prod_n (1 + aV_n)\right] = 1 + \frac{1}{\alpha}(e^{\alpha a} - 1) \tag{170}$$

$$E_{\alpha,0}\left[\prod_n \left(1 + aV_n^2\right)\right] = 1 + \frac{2}{\alpha}(\cosh(\sqrt{\alpha(1-\alpha)a} - 1) \tag{171}$$

Examination of the coefficient of a^k on both sides of (169) shows that (169) amounts to the following identity: for all positive integers k and p,

$$E_{\alpha,0}\left[\sum_{1 \le n_1 < \cdots < n_k} V_{n_1}^p \cdots V_{n_k}^p\right] = \frac{p}{\alpha}\frac{1}{(pk)!}\left(\frac{\alpha ,(p-\alpha)}{,(1-\alpha)}\right)^k \tag{172}$$

This is a special case of formula (177). Taking

$$\theta = 0, n = pk, m_p = k, m_j = 0 \text{ for } j \neq p,$$

in (177) and multiplying both sides by $k!$ yields (172). Also from (177), or by variations of the above argument one can read analogs of (172) and (169) for PD(α, θ) and results for other polynomials. For instance, (167) can be inverted explicitly for $g(v) = 1 + av + bv^2$.

To conclude this section, we record the following analog of Corollary 47 for PD(α, θ) instead of PD$(\alpha, 0)$:

Corollary 48 *For $0 < \alpha < 1$, $\theta > 0$, $\lambda > 0$, and g and $K_g(\alpha, \lambda)$ as in Corollary 47,*

$$\int_0^\infty ds\, e^{-\lambda s} \frac{s^{\theta-1}}{\Gamma(\theta)} E_{\alpha,\theta}\left[\prod_n g(sV_n)\right] = \left(\frac{\Gamma(1-\alpha)}{\alpha K_g(\alpha, \lambda)}\right)^{\theta/\alpha} \tag{173}$$

Proof. This can be obtained from the previous results using formula (43), but we prefer the following derivation from Proposition 20. Replacing $g(v)$ by $e^{v-\lambda v} g(v)$, it suffices to establish the formula for $\lambda = 1$. Let $V_n(T)$ be derived from (τ_s) and $T = \tau(S_{\alpha,\theta})$ as in Proposition 20. By application of that Proposition, $E\left[\prod_n g(V_n(T))\right]$ equals the left side of (173) for $\lambda = 1$. But evaluating this expectation by conditioning on $S_{\alpha,\theta}$ and using (163) yields the right side of (173) for $\lambda = 1$. □

9 Appendix

We mention in this appendix some known results which provide motivation for the definition and study of PD(α, θ) .

9.1 The finite Poisson-Dirichlet distribution

If the convention is made that the beta(a, b) distribution is a unit mass at 1 for $a > 0, b = 0$, then for (α, θ) in the range

$$\alpha = -\kappa \text{ and } \theta = m\kappa \text{ for some } \kappa > 0 \text{ and } m \in \{2, 3, \cdots\} \tag{174}$$

Definition 1 prescribes a joint distribution of a finite sequence $(\tilde{V}_1, \cdots, \tilde{V}_m)$ with $\tilde{V}_i \geq 0$ and $\sum_{i=1}^m \tilde{V}_i = 1$, and the distribution of the corresponding ranked sequence $(V_1, \cdots, V_m, 0, 0, \cdots)$ with $V_1 \geq \cdots V_m \geq 0$ and $\sum_{i=1}^m V_i = 1$ may still be called PD(α, θ) . It is known that for $(\alpha, \theta) = (-\kappa, m\kappa)$ in this range, $(\tilde{V}_1, \cdots, \tilde{V}_m)$ may be constructed as the size-biased permutation of $(W_1, \cdots, W_m)$, where $(W_1, \cdots, W_m)$ has symmetric Dirichlet distribution obtained by setting $W_i = X_i/(X_1 + \cdots + X_n)$ for i.i.d. X_i with *gamma*(κ) *distribution*, so $(V_1, \cdots, V_m)$ can be obtained by ranking $(W_1, \cdots, W_m)$. See Kingman [34], Section 9.6 for a proof and references. As shown by Kingman [33], as $\kappa = -\alpha \downarrow 0$ and $m \uparrow \infty$ for fixed $\theta = m\kappa$, PD(α, θ) converges weakly to PD$(0, \theta)$. It is easily verified that the formulae in this paper which follow directly from Proposition 2, in particular, (5) (51) and (177) hold also for (α, θ) in the range (174). See also Griffiths [23] for some moment formulae for the finite Poisson-Dirichlet distribution in the vein of (50).

9.2 The partition structure derived from PD(α, θ)

In a random sample of size n from a population with random frequencies $(V_1, V_2, \cdots)$, and a vector of non-negative integers $(m_1, \cdots, m_n)$ with $\Sigma i m_i = n$, the probability that there are m_1 species with a single representative in the sample, and m_2 species with two representatives in the sample, and so on, is given by the formula

$$p(m_1, \cdots m_n) = \frac{n!}{\prod_{i=1}^{n} (i!)^{m_i} m_i!} \mu(m_1, \cdots m_n) \tag{175}$$

with

$$\mu(m_1, \cdots m_n) = E\left[\sum \prod_{i=1}^{n} \prod_{j=1}^{m_i} V_{n(i,j)}^{i} \right] \tag{176}$$

where the summation ranges over all choices of distinct $n(i,j)$ with

$$i = 1, \cdots, n; j = 1, \cdots, m_i$$

. See Kingman [34], where the expectation (176) is evaluated for (V_n) with PD$(0, \theta)$ distribution to obtain the formula for $p(m_1, \cdots m_n)$ in this case, which is the *Ewens sampling formula* [18, 19, 20]. Proposition 9 of Pitman [48] gives the generalization of the Ewens formula for PD(α, θ) , which can be stated as follows. For real numbers x and a and non-negative integer m, let

$$[x]_{m,a} = \begin{cases} 1 \text{ for } m = 0 \\ x(x+a)\cdots(x+(m-1)a) \text{ for } m = 1, 2, \ldots \end{cases}$$

and let $[x]_m = [x]_{m,1}$. Note that $[1]_m = m!$.

Proposition 49 [48] *For* (V_n) *with* PD(α, θ) *distribution,* (175) *and* (176) *hold with* $\mu(m_1, \cdots m_n) = \mu_{\alpha,\theta}(m_1, \cdots m_n)$ *given by the formula*

$$\mu_{\alpha,\theta}(m_1, \cdots m_n) = \frac{[\theta + \alpha]_{k-1,\alpha}}{[\theta + 1]_{n-1}} \prod_{j=1}^{n} \left([1-\alpha]_{j-1}\right)^{m_j} \tag{177}$$

See [48, 47, 45, 49] for further developments and applications of this formula. As a consequence of Proposition 49, the urn scheme for generating PD$(0, \theta)$ studied by various authors [8, 26, 28, 14] also admits a two-parameter generalization [48], whose simple form provides another characterization of the two-parameter family [66].

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