Laplace Transforms Related to Excursions of a One-dimensional Diffusion*

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Abstract

Various known expressions in terms of hyperbolic functions for the Laplace transforms of random times related to one-dimensional Brownian motion are derived in a unified way by excursion theory and extended to one-dimensional diffusions.

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1 Introduction

It is well known that the Laplace transforms of many random times derived from a one-dimensional Brownian motion admit simple expressions in terms of hyperbolic functions. This paper offers a unified approach to these results, and presents their generalizations for a one-dimensional diffusion, using Itô's excursion theory. See also Jeanblanc-Pitman-Yor [4] for a survey of related results involving the Feynman-Kac formula for the distribution of an additive functional of Brownian motion, and Borodin-Salminen [1] for a vast array of formulae for the distribution of functionals of a one-dimensional diffusion.

Section 2 presents the basic univariate formulae in a table, along with commentary and proofs. Section 3 shows how the univariate formulae can be combined with independence results from excursion theory to obtain various multivariate Laplace transforms.

2 Univariate Transforms

Let I be a sub-interval of the real line. Let $(P^x, x \in I)$ govern $X = (X_t, t \geq 0)$ as a non-singular diffusion on I. See [3, 11, 1] for background and precise definitions. Assume for simplicity that X is recurrent. Let $0, x \in I$ with $0 \leq x$. Let $\lambda \geq 0$. In each row of the table on page 3, the lefthand entry is the P^x expectation of some functional of the diffusion path, mostly for $x = 0$. The middle entry gives a general expression for this expectation in terms of three *basic functions*:

$g_\lambda(x, 0)$, the λ-potential density,

$s(x)$, the scale function, and

$\phi_\lambda(x)$, the P_0-Laplace transform of $T_x = \inf\{t : X_t = x\}$.

These basic functions are interpreted probabilistically by Rows (1), (3) and (5) of the table. Analytic expressions for these functions, in terms of the semi-group or generator of X, are standard. Explicit formulae for the basic functions are known for many diffusions. In particular, the third column of the table gives formulae derived from the second column in case X is a reflecting Brownian motion (RBM) on $I = [0, \infty)$, in terms of hyperbolic functions of θx, where $\theta = \sqrt{2\lambda}$. All the formulae in the third column were obtained by Knight [5], who also inverted most of these transforms.

Probabilistic quantity	General expression for $0 \leq x$	Expression for RBM with $\theta = \sqrt{2\lambda}$	
$E^x\left[L_{W_\lambda}\right]$	$g_\lambda(x,0)$	$\dfrac{\exp(-\theta x)}{\theta}$	(1)
$E^0\left[e^{-\lambda\tau_\ell}\right]$	$\exp[-\ell/g_\lambda(0,0)]$	$\exp[-\ell\theta]$	(2)
$E^0\left[L_{T_x}\right]$	$s(x)$	x	(3)
$P^0\left[M_{\tau_\ell} \leq x\right]$	$\exp[-\ell/s(x)]$	$\exp(-\ell/x)$	(4)
$E^0\left[e^{-\lambda T_x}\right]$	$\phi_\lambda(x)$	$\dfrac{1}{\cosh(\theta x)}$	(5)
$E^0\left[L_{T_x\wedge W_\lambda}\right]$	$s_\lambda(x) = g_\lambda(0,0) - \phi_\lambda(x) g_\lambda(x,0)$	$\dfrac{\tanh(\theta x)}{\theta}$	(6)
$E^0\left[e^{-\lambda\tau_\ell} 1(M_{\tau_\ell} \leq x)\right]$	$\exp[-\ell/s_\lambda(x)]$	$\exp[-\ell\theta\ \coth(\theta x)]$	(7)
$E^0\left[e^{-\lambda G_x}\right]$	$\dfrac{s_\lambda(x)}{s(x)}$	$\dfrac{\tanh(\theta x)}{\theta x}$	(8)
$E^0\left[e^{-\lambda(T_x - G_x)}\right]$	$\dfrac{\phi_\lambda(x)s(x)}{s_\lambda(x)}$	$\dfrac{\theta x}{\sinh(\theta x)}$	(9)

The following commentary introduces the notation of the table, line by line, and indicates proofs of the formulae by application of Itô's excursion theory.

Row (1). Let $L = (L_t, t \geq 0)$ be a local time process of X at 0. And let W_λ be exponentially distributed with rate λ, independent of X. This row identifies the potential density probabilistically as

$$g_\lambda(x,0) = E^x\left[L_{W_\lambda}\right] = E^x \int_0^\infty \lambda e^{-\lambda t} L(t)dt = E^x \int_0^\infty e^{-\lambda t} L(dt) = c\int_0^\infty e^{-\lambda t} p(t,x,0)dt \tag{11}$$

where $p(t,x,y) = P_x(X_t \in dy)/m(dy)$ is the jointly continuous transition density of X relative to the speed measure m, and c is a constant depending on the normalization of local time and conventions regarding constant factors in the definition of the scale function and speed measure of X. In the third column, when X is RBM, say $X = |B|$ where B is a standard BM, we take L to be the occupation density of B at 0 relative to Lebesgue measure. Then Lévy's equivalence holds: L_t and $|B_t|$ have the same P^0 distribution.

Row (2). Let $(\tau_\ell, \ell \geq 0)$ be the inverse of L. The general expression for the Laplace transform of τ_ℓ is well known for L the local time process of X at 0 for any recurrent point 0 of a strong Markov process X. This formula follows immediately from the probabilistic definition of $g_\lambda(0,0)$ in Row (1), by Itô's excursion theory. Let P_0 govern a Poisson point process N on $(0,\infty)$ with rate λ, independent of X, and mark each excursion of X away from 0 by the times of points of N during the excursion, if any. Then, as explained in Greenwood-Pitman [2] and Rogers-Williams [11], Section VI.53, one obtains a homogeneous Poisson point process of marked excursions on the local time scale. (In case X spends positive Lebesgue time at 0, this process must also count marks between excursions). Let W_λ be the time of the first point of N. Then L_{W_λ} is the time of the first marked excursion on the local time scale, so L_{W_λ} has exponential distribution with rate $1/E(L_{W_\lambda}) = 1/g_\lambda(0,0)$. Thus

$$E^0\left[e^{-\lambda\tau_\ell}\right] = P^0(W_\lambda > \tau_\ell) = P^0(L_{W_\lambda} > \ell) = \exp[-\ell/g_\lambda(0,0)]$$

Analysis of this formula, together with Krein's theory of strings, allowed Knight [7] and Kotani-Watanabe [8] to characterize the Lévy measures of the process of inverse local times $(\tau_\ell, \ell \geq 0)$. In particular, these Lévy measures are absolutely continuous with respect to Lebesgue measure on $(0,\infty)$, and the densities are Laplace transforms. See also Section 6 of Pitman [9].

Row(3). This row defines $s(x)$ for $x > 0$. Note that $1/s(x)$ is the rate per unit local time of excursions from 0 that reach x. So by the Poisson character of the excursion process, and the strong Markov property of X, for $0 < x < y$, given that an excursion reaches x, the chance that it reaches y is

$$P^x(T_y < T_0) = \frac{1/s(x)}{1/s(y)} = \frac{s(y)}{s(x)} \tag{12}$$

That is to say, *the function $s(x)$ serves as a scale function for X on the interval* $[0, \infty]$*, with* $s(0) = 0$.

Row (4). Here $M_t = \max_{0 \le s \le t} X_s$. This is implied by (3) and the Poisson character of excursions on the local time scale, just as (1) implied (2).

Row (5). This row defines $\phi_\lambda(x)$. The evaluation of $\phi_\lambda(x)$ for RBM is made by the following well known argument: for $\theta = \sqrt{2\lambda}$, apply the optional sampling theorem to the martingale $\cosh(\theta|B_t|)\exp(-\lambda t)$ which is the average of the two martingales $\exp(\pm\theta B_t - \lambda t)$.

Row (6). This row defines a new function

$$s_\lambda(x) = E^0\left[L_{T_x \wedge W_\lambda}\right] = E^0[L_{W_\lambda}] - E^0[(L_{W_\lambda} - L_{T_x})1(T_x < W_\lambda)] = g_\lambda(0,0) - \phi_\lambda(x) g_\lambda(x,0) \tag{13}$$

by application of the strong Markov property of X at time T_x, and the definitions of Rows (1) and (3). Substituting the formulae of Rows (1) and (3) for RBM gives the expression $s_\lambda(x) = \theta^{-1}\tanh(\theta x)$ for RBM.

Row (7). This is implied by Row (6), just as Row (1) implies Row (2), and Row (3) implies Row (4). In terms of the Poisson point process of marked excursions, Row (6) shows that $1/s_\lambda(x)$ is the rate of excursions that *either reach x or are marked.* The left and middle entries of Row (7) show two different ways of computing the probability of no such excursions up to local time ℓ.

Row (8). Here G_x is the last zero of X before time T_x. Consider the first excursion that either reaches x or is marked. Compute the probability that this excursion reaches x, first by conditioning on G_x, then from the ratio of Poisson rates $[1/s(x)]/[1/s_\lambda(x)]$, to see that this probability is given by both the left and central entries of Row (8).

Row (9). The Poisson character of the excursion process implies that G_x and $T_x - G_x$ are independent (last exit decomposition). So Row (9) follows from Rows (5) and (8). For X a BM or RBM, the result is implicit in D. Williams' description of the process $(X_{G_x+t}, 0 \le t \le T_x - G_x)$ as a BES(3) process started at 0 and run till it first hits x. (See e.g. Williams [13], formula (67.2) of Ch. II). More generally, if the upper endpoint of the basic interval I on which X is defined is b say, Williams' results show that the P_0 distribution of $(X_{G_x+t}, 0 \leq t \leq T_x - G_x)$ is identical to the $\hat{P}^0$ distribution of $(X_t, 0 \leq t \leq T_x)$ where the family of diffusion laws $(P^x, x \in [0,\infty))$ conditions X to hit b before 0 (the Doob h-transform of X for $h(x) = s(x)$). So Row(9) implies an expression for the $\hat{P}^0$ Laplace transform of T_x:

$$\hat{P}^0[\exp(-\lambda T_x)] = \frac{\phi_\lambda(x)s(x)}{s_\lambda(x)} \tag{14}$$

The generator $\hat{A}$ for this conditioned diffusion is $\hat{A} = s^{-1}As$. Using the standard fact that $1/\phi_\lambda(x)$ is a solution of $Af = \lambda f$, it is easy enough to check that the inverse of the right side of (14) solves $\hat{A}f = \lambda f$. A more careful discussion of boundary behaviour is required to make this observation into an analytic proof of (14). See [4] for related results.

Applications to Brownian Motion. If instead of $X = |B|$ a RBM we consider $X = B$ a BM, for the same normalization of L, the formulae in the table apply for all $x \geq 0$ with

$$g_\lambda(x,0) = \theta^{-1}e^{-\theta x}; \qquad s(x) = 2x; \qquad \phi_\lambda(x) = e^{-\theta x}$$

where $\theta = \sqrt{2\lambda}$. Then $s_\lambda(x) = \theta^{-1}(1 - e^{-2\theta x})$

Applications to other diffusions. Explicit formulae for the basic functions $g_\lambda(x,0)$, $s(x)$ and $\phi_\lambda(x)$ appearing in the table are known for a great many particular diffusion processes of interest, including Bessel and Ornstein-Uhlenbeck processes. See Borodin-Salminen [1].

3 Multivariate Transforms

As noted by Knight [5, 6], the Poisson character of the excursion process implies that $(X_t, 0 \leq t \leq G_x)$ given $L_{T_x} = \ell$ has the same distribution as $(X_t, 0 \leq t \leq \tau_\ell)$ given $(M_{\tau_\ell} < x)$. In particular,

$$E^0\left[\exp(-\alpha G_x)|L_{T_x} = \ell\right] = E^0\left[\exp(-\alpha \tau_\ell)|M_{\tau_\ell} < x\right]. \tag{15}$$

This quantity can be evaluated from Rows (4) and (7) of the table as

$$\exp\left[-\ell\left(\frac{1}{s_\lambda(x)} - \frac{1}{s(x)}\right)\right] \overset{RBM}{=} \exp\left[-\ell(\theta\coth(\theta x) - x^{-1})\right] \tag{16}$$

where the notation $\stackrel{RBM}{=}$ means equality in case X is RBM, and the formula for RBM is due to Knight [5]. Combine this formula with the fact that L_{T_x} has exponential distribution with rate $1/s(x)$ to obtain

$$E^0\left[\exp(-\alpha L_{T_x} - \lambda G_x)\right] = \frac{s_\lambda(x)}{s(x)[1+\alpha s_\lambda(x)]} \quad \stackrel{RBM}{=} \quad \frac{1}{\theta x \coth(\theta x) + \alpha x} \tag{17}$$

Using the independence of (L_{T_x}, G_x) and $T_x - G_x$, and Row (9), this implies

$$E^0\left[\exp(-\alpha L_{T_x} - \lambda T_x)\right] = \frac{\phi_\lambda(x)}{[1+\alpha s_\lambda(x)]} \quad \stackrel{RBM}{=} \quad \frac{\theta}{\theta\cosh(\theta x) + \alpha\sinh(\theta x)} \tag{18}$$

Williams [14] obtained this formula for RBM, and showed how it implies closely related formulae of H.M. Taylor [12].

Let

$$A_t^+ = \int_0^t ds 1(X_s > 0); \qquad A_t^- = \int_0^t ds 1(X_s \le 0) \tag{19}$$

In case 0 is not the lower endpoint of I, it is of interest to consider the joint distribution of $A^+_{T_x}, A^-_{T_x}$ and L_{T_x}. A preliminary calculation, based on the independence of positive and negative excursions and Rows (4) and (7) of the table, yields the formula

$$E^0\left[\exp(-\lambda A^-_{\tau_\ell})\right] = \lim_{x\downarrow 0} E^0\left[\exp(-\lambda\tau_\ell) | M_{\tau_\ell} < x\right] = \exp\left[-\ell\xi_\lambda\right] \tag{20}$$

where

$$\xi_\lambda = \lim_{x\downarrow 0}\left[\frac{1}{s_\lambda(x)} - \frac{1}{s(x)}\right] \quad \stackrel{BM}{=} \quad \theta/2 \tag{21}$$

and $\stackrel{BM}{=}$ means equality in case X is BM, with $\theta = \sqrt{2\lambda}$. Conditioning on $L_{T_x} = \ell$, as before, then integrating out ℓ, yields the formula

$$E^0\left[\exp(-\alpha L_{T_x} - \lambda A^+_{T_x} - \mu A^-_{T_x})\right] = \frac{\phi_\lambda(x)}{1 + (\alpha + \xi_\mu - \xi_\lambda)s_\lambda(x)} \tag{22}$$

$$\stackrel{BM}{=} \quad \frac{\theta}{\theta\cosh(\theta x) + (2\alpha + \sqrt{2\mu})\sinh(\theta x)} \tag{23}$$

The formula for BM was obtained by Pitman-Yor [10] (proof of Theorem 4.2), using martingale calculus.

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