On the relative lengths of excursions derived from a stable subordinator *

by
Jim Pitman and Marc Yor

To appear in Séminaire de Probabilités XXXI

Technical Report No. 469

Department of Statistics
University of California
367 Evans Hall # 3860
Berkeley, CA 94720-3860

August 14, 1996

*Research supported in part by N.S.F. Grant DMS-9404345

Abstract

Results are obtained concerning the distribution of ranked relative lengths of excursions of a recurrent Markov process from a point in its state space whose inverse local time process is a stable subordinator. It is shown that for a large class of random times T the distribution of relative excursion lengths prior to T is the same as if T were a fixed time. It follows that the generalized arc-sine laws of Lamperti extend to such random times T. For some other random times T, absolute continuity relations are obtained which relate the law of the relative lengths at time T to the law at a fixed time.

1 Introduction

Following Lamperti [10], Wendel [24], Kingman [7], Knight [8], Perman-Pitman-Yor [12, 13, 15], consider the sequence

$$V_1(T) \geq V_2(T) \geq \cdots \tag{1}$$

of ranked lengths of component intervals of the set $[0,T]\backslash Z$, where T is a strictly positive random time, and Z is the zero set of a Markov process X started at zero, such as a Brownian motion or Bessel process, for which the inverse $(\tau_s, s \geq 0)$ of the local time process of X at zero is a *stable*(α) *subordinator*, that is an increasing process with stationary independent increments and Lévy measure Λ_α where

$$\Lambda_\alpha(x,\infty) = Cx^{-\alpha} \qquad (x>0) \tag{2}$$

for some constant $C>0$, and $0<\alpha<1$. That is, for $\lambda>0$

$$E[\exp(-\lambda\tau_s)] = \exp(-sK\lambda^\alpha) \text{ where } K = C\Gamma(1-\alpha). \tag{3}$$

It was shown in [15] that for all $t>0$ and $s>0$

$$\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}, \cdots\right) \stackrel{d}{=} \left(\frac{V_1(\tau_s)}{\tau_s}, \frac{V_2(\tau_s)}{\tau_s}, \cdots\right) \tag{4}$$

where $\stackrel{d}{=}$ denotes equality in distribution. Write simply V_n instead of $V_n(1)$, so $(V_1, V_2, \cdots)$ is a convenient notation for a sequence of random variables with the common joint distribution of the sequences displayed in (4) for all $s > 0$ and $t > 0$. The distribution of (V_n) of course depends on α, but we suppress α in the notation. Note that

$$V_1 > V_2 > \cdots > 0 \text{ a.s. and } \textstyle\sum_n V_n = 1 \text{ a.s.} \tag{5}$$

For a detailed account of features of the distribution of (V_n) with a parameter $0 < \alpha < 1$, references to earlier work, and connections with Kingman's [7] Poisson-Dirichlet distribution, see [17]. Our main purpose in this paper is to point out that beyond the fixed times t and inverse local times τ_s featured in (4), there are many more random times T such that

$$\left(\frac{V_1(T)}{T}, \frac{V_2(T)}{T}, \cdots\right) \stackrel{d}{=} (V_1, V_2, \cdots) \tag{6}$$

Definition 1 Call T *admissible*, or to be more precise *admissible for* Z, if (6) holds. Call T *inadmissible* otherwise.

Note that Definition 1 makes sense for any random closed subset Z of $\mathbb{R}^+$, and any $\mathbb{R}^+$-valued random variable T, with $V_n(T)$ defined as the nth longest component interval of $[0,T]\backslash Z$ and $V_n := V_n(1)$. In this paper we obtain some general results which clarify the relation between stability properties of Z and admissibility of various random times T for Z. But for the rest of the introduction we continue to assume that Z is the closure of the range of a stable(α) subordinator.

We showed in [17] by direct calculation that

$$H_m := \inf\{t : V_m(t) \geq 1\} \textit{ is admissible for each } m = 1, 2, \cdots \tag{7}$$

Here we provide a criterion for a random time T to be admissible, which yields a large family of random times, including the times t, τ_s and H_m mentioned above, which are admissible for Z derived from a stable(α) subordinator. Let

$$G_t = \sup(Z \cap [0,t)); \qquad D_t = \inf(Z \cap [t, \infty)) \tag{8}$$

The admissibility of H_m turns out to be intimately connected with the following *sampling property* of Z, established in [15], which finds several applications in this paper:

$$P(1 - G_1 = V_n | V_1, V_2, \cdots) = V_n \qquad (n = 1, 2, \cdots) \tag{9}$$

See [14, 18] for further discussion of this property and related results.

The rest of this paper is organized as follows. The main results for the range of a stable subordinator are presented in Section 2 and proved in Section 3. Besides finding times that are admissible, we show for some inadmissible random times T, in particular for $T = G_t$ and $T = D_t$ for a fixed time t, that the distribution of the sequence on the left side of (6) has a simple density relative to that of $(V_1, V_2, \cdots)$. In Section 4 we relate our study of admissible times to the generalized arc-sine laws of Lamperti [9, 10], studied also in [2, 15, 23]. In particular, we describe the distribution of time spent positive by a skew Bessel process or skew Bessel bridge.

2 Results for a Stable Subordinator

Throughout this section, let $0 < \alpha < 1$, and let E_α denote expectation with respect to a probability distribution P_α which governs $(\tau_s, s \geq 0)$ as a stable(α) subordinator, and let Z be the closure of the range of (τ_s). Let $(S_t, t \geq 0)$ denote the continuous *local time process* defined by $S_t = \inf\{s : \tau_s > t\}$. While many approximations of local time are known [4], a useful one in the present setting is the following:

Proposition 2 *For each $t > 0$,*

$$n^{1/\alpha} V_n(t) \to (C S_t)^{1/\alpha} \textit{almost surely } (P_\alpha) \textit{ as } n \to \infty. \tag{10}$$

where the limit holds uniformly in $0 \leq t \leq t_0$ almost surely (P_α) for every $t_0 > 0$, and also in pth mean for every $p > 0$.

Proof. The convergence both a.s. and in pth mean for a fixed $t > 0$ is established in Proposition 10 of [17]. As observed by Kingman [7], (10) holds almost surely with the random time τ_s substituted instead of the fixed time t, and $S_{\tau_s} = s$ instead of S_t. Since $(V_n(t), t \geq 0)$ is an increasing process in t for each n, and $(S_t, t \geq 0)$ is a continuous increasing process, the claimed almost sure convergence can be deduced by a standard argument. See for instance Lemma 2.5 of [5]. □

2.1 Admissible Times

Proposition 3 *Given $c_n \geq 0$ with $\sup_n c_n < \infty$ and $c \geq 0$, let*

$$A_t := \sum_n c_n V_n(t) + c S_t^{1/\alpha} \tag{11}$$

and for $u > 0$ let

$$\alpha_u := \inf\{t : A_t > u\} \tag{12}$$

Then α_u is an admissible time.

Proposition 2 has the following immediate corollary:

Corollary 4 *If T is admissible then*

$$\left(\frac{S_T}{T^\alpha}, \frac{V_1(T)}{T}, \frac{V_2(T)}{T}, \cdots\right) \stackrel{d}{=} (S_1, V_1, V_2, \cdots) \tag{13}$$

where

$$S_1 := C^{-1} \lim_n n V_n^\alpha \text{ almost surely } (P_\alpha) \text{ and in } p\text{th mean for all } p > 0 \tag{14}$$

2.2 Inadmissible Times

Corollary 4 implies that if T is an admissible time such that $P_\alpha(G_T < T) > 0$, then G_T is not admissible. Indeed

$$\frac{S_{G_T}}{G_T^\alpha} = \frac{S_T}{G_T^\alpha} \geq \frac{S_T}{T^\alpha}$$

and the inequality is strict on the event $(G_T < T)$. So S_{G_T}/G_T^α cannot have the same distribution as S_T/T^α if $P_\alpha(G_T < T) > 0$. Similar remarks apply to D_T. For a constant time t, the sequence $\left(\frac{V_1(G_t)}{G_t}, \frac{V_2(G_t)}{G_t}, \cdots\right)$ is independent of G_t with the same distribution as the sequence of ranked lengths of excursion intervals of the corresponding bridge of length 1. This follows from the fact (easily verified using the invariance of Bessel processes under time inversion [22]) that if $(R_t, t \geq 0)$ is a Bessel process of dimension $2 - 2\alpha$ starting at 0, then $(G_t^{-1/2} R_{uG_t}, 0 \leq u \leq 1)$ is a standard Bessel bridge of the same dimension independent of G_t. From Theorem 5.3 of [15], there is the following density formula relative to the distribution of $(V_1, V_2, \cdots)$: for all non-negative product measurable f

$$E_\alpha \left[f\left(\frac{V_1(G_t)}{G_t}, \frac{V_2(G_t)}{G_t}, \cdots \right) \right] = \frac{E_\alpha \left[S_1 f(V_1, V_2, \cdots) \right]}{E_\alpha(S_1)} \tag{15}$$

Let N_t be the rank of the meander length $t - G_t$ in the sequence of excursion lengths $V_1(t) > V_2(t) > \cdots$, so $t - G_t = V_{N_t}(t)$. Formula (9) amounts to the formula

$$E_\alpha \left[1(N_t = n)\, f\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t} \cdots \right) \right] = E_\alpha[V_n f(V_1, V_2, \cdots)] \tag{16}$$

for all $n = 1, 2 \cdots$ and all non-negative product measurable functions f. Consider now N_{D_t}, the rank of the excursion length $D_t - G_t$ straddling t in the sequence of complete excursion lengths $V_1(D_t) > V_2(D_t) > \cdots$. So $N_t - 1$ is the number of excursions completed by time t whose lengths exceed $t - G_t$, and $N_{D_t} - 1$ is the smaller number of such excursions whose lengths exceed $D_t - G_t$.

Proposition 5 *For each $t > 0$ and $n = 1, 2, \cdots$,*

$$E_\alpha \left[1(N_{D_t} = n)\, f\left(\frac{V_1(D_t)}{D_t}, \frac{V_2(D_t)}{D_t} \cdots \right) \right] = E_\alpha[-\alpha \log(1 - V_n) f(V_1, V_2, \cdots)] \tag{17}$$

Immediately from Proposition 5, we draw the following consequences. First, summing over n gives

$$E_\alpha \left[f\left(\frac{V_1(D_t)}{D_t}, \frac{V_2(D_t)}{D_t} \cdots \right) \right] = E_\alpha \left[\left(-\sum_n \alpha \log(1 - V_n) \right) f(V_1, V_2, \cdots) \right] \tag{18}$$

which is the analog of (15) for D_t instead of G_t. Next, an analog of (9) for D_t instead of t can be read from (17) as follows: for each $n = 1, 2, \cdots$

$$P_\alpha \left(D_t - G_t = V_n(D_t) \,\middle|\, \frac{V_m(D_t)}{D_t} = u_m, m = 1, 2, \cdots \right) = \frac{\log(1 - u_n)}{\sum_m \log(1 - u_m)} \tag{19}$$

Note the remarkable fact that, just as in (9), the conditional distribution does not depend on α.

Finally, by taking $f = 1$ in (17), we obtain the formula

$$P_\alpha(N_{D_t} = n) = E_\alpha[-\alpha \log(1 - V_n)] \tag{20}$$

As noted in [17], combined with (14) and (16) this allows the asymptotic evaluations as $n \to \infty$:

$$P_\alpha(N_{D_t} = n) \sim \alpha P_\alpha(N_t = n) \sim \frac{\alpha \Gamma(\frac{1}{\alpha} + 1)}{\Gamma(1-\alpha)^{1/\alpha}} \frac{1}{n^{1/\alpha}} \tag{21}$$

where $a(n) \sim b(n)$ means $a(n)/b(n) \to 1$ as $n \to \infty$. See [19, 17] for integral expressions for the distributions of N_t and N_{D_t}, and some numerical values.

In (15) and (17) we have described the law of $(V_1(T)/T, V_2(T)/T, \ldots)$ for $T = G_t$ and for $T = D_t$ by a change of measure relative to the law of this random vector for a fixed time T. By similar arguments we obtain change of measure formulae for $T = G_{H_n}$ and $T = D_{H_n}$. We now give these descriptions for $n = 1$.

Proposition 6 *For each non-negative product measurable function f,*

$$E_\alpha \left[f\left(\frac{V_1(G_{H_1})}{G_{H_1}}, \frac{V_2(G_{H_1})}{G_{H_1}}, \ldots \right) \right] = E_\alpha \left[\left(\frac{S_1}{V_1^\alpha} \right) f(V_1, V_2, \ldots) \right] \tag{22}$$

$$E_\alpha \left[f\left(\frac{V_1(D_{H_1})}{D_{H_1}}, \frac{V_2(D_{H_1})}{D_{H_1}}, \ldots \right) \right] = E_\alpha \left[\left(\alpha \log \frac{V_1}{V_2} \right) f(V_1, V_2, \ldots) \right] \tag{23}$$

As checks, we recall from [17, Props. 10 and 8] that under P_α the distribution of S_1/V_1^α is standard exponential, whereas the distribution of V_2/V_1 is beta$(\alpha, 1)$. Therefore, both S_1/V_1^α and $\alpha \log(V_1/V_2)$ are random variables whose P_α expectation equals 1, as implied by (22) and (23) for $f = 1$.

3 Proofs

3.1 Admissible times

The foundation for the proof of Proposition 3 is a scaling argument which may prove useful in other contexts. The following theorem presents the conclusion of this argument in a fairly general setting.

Recall that a real or vector-valued process $(X_t, t > 0)$ is called *β-self-similar* for some $\beta \in \mathbb{R}$ if for every $c > 0$

$$(X_{ct}, t > 0) \stackrel{d}{=} (c^\beta X_t, t > 0) \tag{24}$$

See [20] for a survey of the literature of these processes. Note that (X_t) is β-self-similar iff the process (Y_t) defined by $Y_t = t^{-\beta} X_t$ is 0-self-similar, that is to say, for every $c > 0$

$$(Y_{ct}, t > 0) \stackrel{d}{=} (Y_t, t > 0) \tag{25}$$

This definition of 0-self-similarity makes sense even for Y with values in an abstract measurable space where there is no notion of scalar multiplication. Suppose now that X is viewed as a measurable map from the basic probability space to a suitable path space $(S, \mathcal{S})$, e.g. $S = C[0, \infty)$ and $\mathcal{S}$ the σ-field generated by coordinate maps, assuming X has continuous paths. Suppose (X_t) is β-self-similar. Let $(\mathbf{X}_t, t > 0)$ denote the path valued process defined by letting $\mathbf{X}_t$ be the rescaling of $\mathbf{X}$ that maps time t to time 1, that is

$$\mathbf{X}_t(s) = t^{-\beta} X_{st} \qquad (s \geq 0) \tag{26}$$

Then it is easily verified that $(\mathbf{X}_t, t > 0)$ is 0-self-similar.

It is this kind of 0-self-similar process which we have in mind for applications of the following theorem.

Theorem 7 *Let $(\mathbf{X}_t, t > 0)$ be a jointly measurable 0-self-similar process with values in an arbitrary measurable space $(S, \mathcal{S})$. Let $\theta_s = \Theta(\mathbf{X}_s)$ for a non-negative $\mathcal{S}$-measurable function Θ defined on S, let*

$$A_t = \int_0^t \theta_s ds, \qquad (t \geq 0) \tag{27}$$

$$\alpha_u = \inf\{t : A_t > u\} \qquad (u \geq 0) \tag{28}$$

Suppose that $0 < A_1 < \infty$ a.s. Then $0 < \alpha_u < \infty$ a.s. for every $u > 0$, and for all non-negative product measurable ψ defined on $S \times [0, \infty)$

$$E\left[\psi(\mathbf{X}_{\alpha_1}, 1/\alpha_1)\right] = E\left[\psi(\mathbf{X}_1, A_1)\,\frac{\theta_1}{A_1}\right] \tag{29}$$

Remarks. According to (29), the law of $(\mathbf{X}_{\alpha_1}, 1/\alpha_1)$ on the product space $S \times [0, \infty)$ is absolutely continuous with respect to that of $(\mathbf{X}_1, A_1)$, with Radon-Nikodym density g defined by

$$g(\mathbf{X}_1, A_1) = \frac{E\left[\theta_1 | \mathbf{X}_1, A_1\right]}{A_1}$$

It follows that for an arbitrary product measurable map Ψ whose range can be any measurable space,

$$\Psi(\mathbf{X}_{\alpha_1}, 1/\alpha_1) \stackrel{d}{=} \Psi(\mathbf{X}_1, A_1) \text{ iff } E\left[\left.\frac{\theta_1}{A_1}\right| \Psi(\mathbf{X}_1, A_1)\right] = 1 \tag{30}$$

For $\Psi(\mathbf{x}, a)$ such that a can be recovered as a measurable function of $\Psi(\mathbf{x}, a)$, condition (30) reduces to

$$E\left[\theta_1 | \Psi(\mathbf{X}_1, A_1)\right] = A_1 \tag{31}$$

In particular, since it follows immediately from the 0-self-similarity of the process (θ_s) that

$$1/\alpha_1 \stackrel{d}{=} A_1 \tag{32}$$

we learn from (30) that

$$E[\theta_1 | A_1] = A_1 \tag{33}$$

Taking $\mathbf{X}_t = \theta_t$ shows that the identity (33) holds for an arbitrary non-negative 0-self-similar process (θ_t) and $A_1 = \int_0^1 \theta_s ds$. See [16, 18] for further developments and applications of this identity. Formula (29) is an abstract version of a result of Yor [26] in the case that $(\mathbf{X}_t)$ is the path-valued process derived by the scaling transformation (26) starting from a Brownian motion (X_t). A consequence of (29) is the following variation of the result of [26] for Brownian motion.

Corollary 8 *Let $(X_t, t \geq 0)$ a β-self-similar process and let $(\theta_t, t \geq 0)$ be such that for each $c > 0$*

$$(X_{ct}, \theta_{ct}; t \geq 0) \stackrel{d}{=} (c^\beta X_t, \theta_t; t \geq 0) \tag{34}$$

Then, with A_1 and α_1 defined as in (27) *and* (28)*, for all non-negative measurable functions F defined on the path space*

$$E\left[F\left(\frac{X_{t\alpha_1}}{\alpha_1^\beta}; t \geq 0\right)\right] = E\left[\frac{\theta_1}{A_1} F(X_t; t \geq 0)\right] \tag{35}$$

Proof of Theorem 7. The following proof of (29) is a simple adaptation of the argument in [26]. Since the bivariate process $((\mathbf{X}_t, \frac{A_t}{t}), t \geq 0)$ is also 0-self-similar, it suffices to prove (29) for ψ of the form $\psi(\mathbf{x}, a) = \phi(\mathbf{x})$ for an arbitrary non-negative $\mathcal{S}$-measurable function ϕ. For h a non-negative Borel function with $\int_0^\infty s^{-1}h(s)ds < \infty$, consider the quantity

$$Q = \int_0^\infty ds\, h(s) E\left[\frac{\theta_s}{A_s}\phi(\mathbf{X}_s)\right] \tag{36}$$

On the one hand, the assumption that $(\mathbf{X}_s)$ is 0-self-similar and the definitions of θ_s and A_s imply that $((\theta_s, A_s/s, \mathbf{X}_s), s > 0)$ is 0-self-similar. So $(\theta_s, A_s, \mathbf{X}_s) \stackrel{d}{=} (\theta_1, sA_1, \mathbf{X}_1)$ and we can compute

$$Q = \left(\int_0^\infty \frac{ds}{s} h(s)\right) E\left[\frac{\theta_1}{A_1}\phi(\mathbf{X}_1)\right] \tag{37}$$

On the other hand, using Fubini, a time change, and using scaling again to see that $(\alpha_t, \mathbf{X}_{\alpha_t}) \stackrel{d}{=} (t\alpha_1, \mathbf{X}_{\alpha_1})$, we can compute

$$\begin{aligned} Q &= E\left[\int_0^\infty \frac{dt}{t} h(\alpha_t)\phi(\mathbf{X}_{\alpha_t})\right] \\ &= E\left[\int_0^\infty \frac{dt}{t} h(t\alpha_1)\phi(\mathbf{X}_{\alpha_1})\right] \\ &= \left(\int_0^\infty \frac{ds}{s} h(s)\right) E\left[\phi(\mathbf{X}_{\alpha_1})\right] \end{aligned} \tag{38}$$

Comparison of (38) with (37) yields (29) for $\psi(\mathbf{x}, a) = \phi(\mathbf{x})$, as was to be proved. $\square$

Proposition 9 *Suppose that Z is the closure of the random set of zeros of a β-self-similar process $(X_t, t \geq 0)$, and assume that the Lebesgue measure of Z is 0 almost surely. Let $V_1(t) \geq V_2(t) \geq \cdots$ be the ranked lengths of the component intervals of $[0, t]\backslash Z$, and put $V_n = V_n(1)$. Let $\mathbf{X}_t$ be the 0-self-similar path valued process defined as in (26) by $\mathbf{X}_t(s) = t^{-\beta}X_{st}, s \geq 0$, let $\theta_s = \Theta(\mathbf{X}_s)$ for a non-negative $\mathcal{S}$-measurable function Θ, and for $t \geq 0$ and*

$u \geq 0$, let $A_t = \int_0^t \theta_s ds$, assume that $0 < A_1 < \infty$ almost surely, and let $\alpha_u = \inf\{t : A_t > u\}$. Then

$$E\left[F\left(\frac{V_n(\alpha_1)}{\alpha_1}, n \geq 1\right)\right] = E\left[F(V_n, n \geq 1)\frac{\theta_1}{A_1}\right] \tag{39}$$

for all non-negative product measurable functions F. Consequently, α_1 is admissible, meaning

$$\left(\frac{V_1(\alpha_1)}{\alpha_1}, \frac{V_2(\alpha_1)}{\alpha_1}, \cdots\right) \stackrel{d}{=} (V_1, V_2, \cdots) \tag{40}$$

if and only if

$$E\left[\frac{\theta_1}{A_1}\,\middle|\, V_1, V_2, \cdots\right] = 1. \tag{41}$$

Proof. Since for each n, and every $t > 0$, $V_n(t)/t = f_n(\mathbf{X}_t)$ for a measurable function f_n which does not depend on t, formula (39) follows immediately from the previous theorem. □

Note that in case A_1 is a measurable function of $(V_n, n \geq 1)$, the condition (41) becomes

$$E[\theta_1 | V_1, V_2, \cdots] = A_1. \tag{42}$$

Corollary 10 *Let A_t be the time spent positive by a standard Brownian motion B up to time t, so α_1 is the first instant that B has spent time 1 positive. Then α_1 is admissible for the zero set of B.*

Proof. We show that (41) holds. Clearly, it suffices to show that

$$E\left[\theta_1 \,\middle|\, A_1, V_1, V_2, \cdots\right] = A_1 \tag{43}$$

where $\theta_1 = 1(B_1 > 0)$. Let ε_n be the indicator of the event that B is positive on the interval whose length is V_n. Since the V_n are a.s. all distinct, there are a.s. no quibbles about the definition of the ε_n. By Itô's excursion theory, the ε_n are independent Bernoulli($\frac{1}{2}$) variables, independent of $(G_1, V_1, V_2, V_3, \cdots)$, and by definition

$$\theta_1 = \sum_n \varepsilon_n 1(1 - G_1 = V_n) \text{ and } A_1 = \sum_n \varepsilon_n V_n$$

so we have, by the sampling property (9),

$$E(\theta_1|\varepsilon_1,\varepsilon_2,\cdots,V_1,V_2,\cdots) = \sum_n \varepsilon_n P(1-G_1 = V_n|V_1,V_2,\cdots)$$

$$= \sum_n \varepsilon_n V_n = A_1$$

and (43) follows. □

Remark 11 It is clear from the above proof that the conclusion of Corollary 10 holds just as well for B a skew Brownian motion or a skew Bessel process, as discussed in Section 4.

Remark 12 As a companion to (43) we note that the sampling property (9) and [25, Exercise 3.4] imply that if $V_1, V_2, \ldots$ are the ranked interval lengths generated by the zero set of a Bessel process $(R_t, 0 \le t \le 1)$ of dimension $2-2\alpha$ started at $R_0 = 0$ then for $x > 0$

$$P(R_1 \in dx \,|\, V_1, V_2, \ldots) = x\,dx \,\sum_{n=1}^{\infty} \exp\left(-\frac{x^2}{2V_n}\right)$$

Corollary 13 *In the setting of Proposition 9, the random time* $H_n := \inf\{t : V_n(t) \ge 1\}$ *is admissible for* Z *iff*

$$P(1-G_1 = V_n|V_1,V_2,\cdots) = V_n \tag{44}$$

Proof. Observe that for each n the process

$$\theta_s := 1(s - G_s = V_n(s)) \tag{45}$$

is of the form $\theta_s = \Theta(\mathbf{X}_s)$ required in Theorem 7 and Proposition 9. Moreover, as observed in [18], the corresponding A_t is just

$$V_n(t) = \int_0^t ds\, 1(s - G_s = V_n(s)) \tag{46}$$

so the corresponding α_1 equals H_n as defined in (7). □

In particular, H_n is admissible for every n iff (44) holds for every n. We then say that Z has the *sampling property*. For Z the range of a stable(α) subordinator, the sampling property of Z was established in [15] while the admissibility of H_n for all n was shown in [17]. Neither of these results seems obvious without some calculation. In [18] we give examples of various 0-self-similar sets Z, some with and some without the sampling property. It would be interesting to characterize all 0-self-similar sets Z with the sampling property, but we have no idea how to do this.

Proof of Proposition 3. Note first that if (T_n) is a sequence of admissible times, and T_n converges in probability as $n \to \infty$ to T with $T > 0$ a.s., then T is admissible. By this observation and Proposition 2, it suffices to prove Proposition 3 for

$$A_t = \sum_{k=1}^{p} c_k V_k(t)$$

In this case we have from (46)

$$\theta_t = \sum_{k=1}^{p} c_k 1(t - G_t = V_k(t))$$

so the sampling property and linearity of conditional expectations imply (42). □

The class of admissible times is preserved under certain homogeneous transformations described in the following proposition.

Proposition 14 *In the setting of Proposition 9, with Z the closure of the random set of zeros of a β-self-similar process $(X_t, t \geq 0)$, the Lebesgue measure of Z equal to 0 almost surely, and $\mathbf{X}_t$ the 0-self-similar path valued process defined by $\mathbf{X}_t(s) = t^{-\beta} X_{st}, s \geq 0$, suppose for each $1 \leq j \leq k$ that $\theta_s^{(j)} = \Theta^{(j)}(\mathbf{X}_s)$ for a non-negative $\mathcal{S}$-measurable function $\Theta^{(j)}$, and for $t \geq 0$ and $u \geq 0$ let $A_t^{(j)} = \int_0^t \theta_s^{(j)} ds$ be such that $0 < A_1^{(j)} < \infty$ almost surely, and define $\alpha_u^{(j)} = \inf\{t : A_t^{(j)} > u\}$. Suppose further for each $1 \leq j \leq k$ that $A_1^{(j)}$ is $\mathcal{V}$-measurable, where $\mathcal{V}$ is the $\sigma-$field generated by $V_1, V_2, \cdots$, and that $\alpha_1^{(j)}$ is admissible for Z. Let $f : \mathbb{R}_+^k \to \mathbb{R}_+$ be an increasing function in each variable such that*

$$f(cx_1, cx_2, \cdots, cx_k) = cf(x_1, x_2, \cdots, x_k) \tag{47}$$

and f is differentiable on $(0, \infty)^k$, and let $A_t := f(A_t^{(1)}, \cdots, A_t^{(k)})$.
Then $\alpha_1 := \inf\{t : A_t > 1\}$ is admissible.

Proof. By calculus $A_t = \int_0^t \theta_s ds$ where

$$\theta_s = \sum_{i=1}^{k} f_i'(A_s^{(1)}, \cdots, A_s^{(k)})\theta_s^{(i)}$$

Thus we can compute

$$\begin{aligned} E[\theta_1|\mathcal{V}] &= \sum_{i=1}^{k} f_i'(A_1^{(1)}, \cdots, A_1^{(k)}) E[\theta_1^{(i)} \mid \mathcal{V}] \\ &= \sum_{i=1}^{k} f_i'(A_1^{(1)}, \cdots, A_1^{(k)}) A_1^{(i)} \end{aligned}$$

by (42). But, from the hypotheses on f we deduce that $\sum_{i=1}^{k} f_i'(x_1, \cdots, x_k)x_i = f(x_1, \cdots, x_k)$ so we obtain $E[\theta_1 \mid \mathcal{V}] = A_1$, as in (42). Therefore, α_1 is admissible. □

Note that the class of functions f considered above is much larger than the class of functions of the form $f(x) = \sum_{i=1}^{k} c_i x_i$. For instance, one can take

$$f_p(x_1, \cdots, x_k) = \left(\sum_{i=1}^{k} (c_i x_i)^p \right)^{1/p}$$

for $p > 0$ and positive constants c_i. By passage to the limit, it can be deduced that the conclusion of Proposition 14 also holds for

$$f(x_1, \cdots, x_k) = \max_{1 \le i \le k} x_i$$

3.2 The lengths at time D_t

Proof of Proposition 5. Let $\mathbf{V}(T) = (V_1(T), V_2(T), \cdots)$ denote the sequence of ranked lengths of component intervals of $[0,T]\backslash Z$ for Z the closed range of a stable subordinator (τ_s). By scaling, the distribution of $\mathbf{V}(D_t)/D_t$ for fixed $t > 0$ does not depend on t. So let us write simply D for D_1 and G for G_1, and compute the law of $\mathbf{V}(D)/D$. Recall that the sequence $\mathbf{V}(1)$ contains the term $1 - G$ as $1 - G = V_N(1)$ for a random index N. The sequence $\mathbf{V}(D)$ is derived from $\mathbf{V}(1)$ by first substituting $D - G$ for this term, then reranking. Let (S_t) be the local time inverse of (τ_s). Let $S = S_1$. So

$S^{-1/\alpha} \stackrel{d}{=} \tau_1$. Consider the three point processes N_1, N_G, and N_D on $(0,\infty)$ defined as follows for $T = 1$, $T = G$ or $T = D$:

$$N_T(\cdot) = \sum_n 1(S^{-1/\alpha} V_n(T) \in \cdot)$$

Let $X := S^{-1/\alpha}(1-G)$ and $Y := S^{-1/\alpha}(D-G)$. Then

$$N_G = N_1 - \delta_X = N_D - \delta_Y$$

where $\delta_W(\cdot) = 1(W \in \cdot)$. According to Theorems 2.1 and 1.2 of [15], P_α governs N_1 as a Poisson random measure with intensity measure Λ_α on $(0,\infty)$ where Λ_α is the stable(α) Lévy measure, and given N_1 the point X is a size-biased pick from the points of N_1. That is to say

$$P_\alpha(N_G \in dn, X \in dx) = \frac{x}{\Sigma n + x} P_\alpha(N_1 \in dn)\Lambda_\alpha(dx) \tag{48}$$

where for a counting measure n on $(0,\infty)$, $\Sigma n = \int_0^\infty x n(dx)$ is the sum of locations of the points of n. Let

$$R := \frac{Y}{X} = \frac{D-G}{1-G}$$

From asymptotic renewal theory [3], or by the last exit decomposition at time G, there is the formula

$$P_\alpha(G \in dx, D \in dy) = \frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{x^{\alpha-1}}{(y-x)^{\alpha+1}}\, dx\, dy \qquad (0 < x < 1 < y < \infty) \tag{49}$$

which implies that G and R are independent, with

$$P_\alpha(R \in dr) = \frac{\alpha}{r^{\alpha+1}}\, dr \qquad (r > 1) \tag{50}$$

The last exit decomposition at time G and scaling imply further that G, N_G and R are mutually independent. Since S is a measurable function of G and N_G, so is X, and we can compute for $y > x$

$$P_\alpha(Y \in dy | N_G, X = x) = P_\alpha(XR \in dy | N_G, X = x) = P_\alpha(xR \in dy)$$

$$= P_\alpha(R \in \frac{dy}{x}) = \alpha \left(\frac{x}{y}\right)^{\alpha+1} \frac{dy}{x}$$

and hence

$$
\begin{aligned}
P_\alpha(N_G \in dn, Y \in dy) &= \int_0^y P_\alpha(N_G \in dn, X \in dx, Y \in dy) \\
&= \left(\int_0^y \frac{x}{\Sigma n + x}\Lambda_\alpha(dx) P_\alpha(Y \in dy | N_G, X = x)\right) P_\alpha(N_1 \in dn) \\
&= \left(\int_0^y \frac{x}{\Sigma n + x}\frac{C\,\alpha\, dx}{x^{\alpha+1}}\alpha\left(\frac{x}{y}\right)^{\alpha+1}\frac{1}{x}\right) P_\alpha(N_1 \in dn)\, dy \\
&= \alpha\left(\int_0^y \frac{dx}{\Sigma n + x}\right) P_\alpha(N_1 \in dn)\Lambda_\alpha(dy) \\
&= \alpha \log\left(\frac{\Sigma n + y}{\Sigma n}\right) P_\alpha(N_1 \in dn)\Lambda_\alpha(dy)
\end{aligned}
$$

That is to say

$$
P_\alpha(N_G \in dn, Y \in dy) = \rho(y | n + \delta_y) P_\alpha(N_1 \in dn)\Lambda_\alpha(dy) \tag{51}
$$

where for a counting measure m

$$
\rho(y|m) = \alpha \log\left(\frac{\Sigma m}{\Sigma m - y}\right)
$$

Since $N_D = N_G + \delta_Y$ and N_1 is a Poisson measure with intensity Λ_α, the Palm formula of [15, Lemma 2.2] shows that (51) can be recast as

$$
P_\alpha(N_D \in dm, Y \in dy) = \rho(y|m) P_\alpha(N_1 \in dm)\Lambda_\alpha(dy) \tag{52}
$$

which implies that

$$
P_\alpha(N_D \in dm) = \rho(m) P_\alpha(N_1 \in dm) \tag{53}
$$

where

$$
\rho(m) = \int \rho(y|m) m(dy) = \alpha \sum_{y:m\{y\}=1} \log\left(\frac{\Sigma m}{\Sigma m - y}\right).
$$

Now

$$
\frac{\mathbf{V}(T)}{T} = \frac{S^{-1/\alpha}\mathbf{V}(T)}{S^{-1/\alpha} T}
$$

Since for $T=1$ and $T=D$, both $S^{-1/\alpha}\mathbf{V}(T)$ and $S^{-1/\alpha}T = \sum_n S^{-1/\alpha}V_n(T)$ are measurable functions of N_T, so is $\mathbf{V}(T)/T$. Since also

$$\rho(N_T) = \alpha \sum_i \log\left(\frac{T}{T - V_i(T)}\right) = -\alpha \sum_i \log\left(1 - \frac{V_i(T)}{T}\right) \tag{54}$$

is a function of $\mathbf{V}(T)/T$, a change of variables in (53) yields (18). A similar manipulation of (52) yields (17). □

As noted in [17], formula (9) implies that for every non-negative measurable function f defined on $[0,1]$,

$$E_\alpha\left[\sum_n f(V_n)\right] = E_\alpha\left[\frac{f(1-G_1)}{(1-G_1)}\right] = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 du f(u) u^{-\alpha-1}(1-u)^{\alpha-1} \tag{55}$$

where the last expression is obtained from the beta$(\alpha, 1-\alpha)$ density of G_1. The consequence of (18), that

$$E_\alpha\left(-\alpha\sum_n \log(1-V_n)\right) = 1$$

therefore amounts to the formula

$$\frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 du(-\log(1-u))u^{-\alpha-1}(1-u)^{\alpha-1} = 1 \tag{56}$$

This identity can be checked directly as follows. Expanding

$$-\log(1-u) = u + \frac{u^2}{2} + \frac{u^3}{3} + \cdots$$

allows the left side of (56) to be evaluated as

$$\frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(B(1-\alpha,\alpha) + \frac{1}{2}B(2-\alpha,\alpha) + \frac{1}{3}B(3-\alpha,\alpha) + \cdots\right)$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, so (56) reduces to

$$\alpha\left(1 + \frac{1-\alpha}{2!} + \frac{(1-\alpha)(2-\alpha)}{3!} + \cdots\right) = 1$$

which can be seen by letting $x \uparrow 1$ in the formula

$$1-(1-x)^{\alpha}=\alpha x+\alpha(1-\alpha)\frac{x^2}{2!}+\alpha(1-\alpha)(2-\alpha)\frac{x^3}{3!}+\cdots \tag{57}$$

obtained from the binomial expansion of $(1-x)^\alpha$. See [14] for an interpretation in terms of a stable(α) subordinator of the discrete distribution with the generating function (57).

A number of variations of the identity (56) can be obtained as follows. Since G_1 has beta$(\alpha, 1-\alpha)$ distribution, if T is an independent exponential variable, then TG_1 has gamma(α) distribution. Therefore, for $\lambda > -1$,

$$E_\alpha\left[\frac{1}{1+\lambda G_1}\right]=\int_0^\infty dt\, e^{-t}E_\alpha(e^{-t\lambda G_1})=E_\alpha[\exp(-\lambda T G_1)]=(1+\lambda)^{-\alpha} \tag{58}$$

Take $\lambda=(1-x)/x$ in (58) to obtain

$$E_\alpha\left[(x+(1-x)G_1)^{-1}\right]=x^{\alpha-1} \qquad (0<\alpha<1, x>0). \tag{59}$$

Integration of (59) with respect to dx over $0<x<a$ yields the formula

$$E_\alpha\left[\frac{1}{1-G_1}\log\left(1+\frac{a(1-G_1)}{G_1}\right)\right]=\frac{a^\alpha}{\alpha} \tag{60}$$

which reduces to (56) for $a=1$. For later reference, we note also the following elementary formula. For an arbitrary non-negative Borel f:

$$E_\alpha\left[\frac{1}{1-G_1}f\left(\frac{1-G_1}{G_1}\right)\right]=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \frac{dv}{v^{\alpha+1}}f(v) \tag{61}$$

3.3 The lengths at times G_{H_1} and D_{H_1}

In this section, we prove Proposition 6. We can assume that Z is the zero set of $\rho := (\rho(u), u \ge 0)$ where under P_α the process ρ is a Bessel process of dimension $2-2\alpha$ started at $\rho(0)=0$. Let π denote the Bessel bridge of dimension $2-2\alpha$ defined by $\pi_u := \rho(uG_1)/\sqrt{G_1}, 0 \le u \le 1$ and let $\tilde\rho$ be the process defined by $\tilde\rho_u := \rho(uG_{H_1})/\sqrt{G_{H_1}}, 0\le u\le 1$.

Proof of (22). This formula is a consequence of (15) and the following absolute continuity relationship between the laws of π and $\tilde\rho$ on $C[0,1]$: for every measurable function $F : C[0,1] \to \mathbb{R}^+$

$$E_\alpha[F(\tilde\rho)]=\gamma_\alpha E_\alpha[(V_1(\pi))^{-\alpha}F(\pi)] \tag{62}$$

where $V_1(\pi)$ denotes the longest excursion interval of the bridge π and

$$\gamma_\alpha := 1/E_\alpha[(V_1(\pi))^{-\alpha}] = E_\alpha[(1-G_1)^\alpha] = \frac{1}{\alpha\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin(\pi\alpha)}{\pi\alpha} \quad (63)$$

Formula (62) is a consequence of the following identity, which we obtain from Corollary 8 with the help of (46):

$$E_\alpha\left[F\left(\frac{\rho(uH_1)}{\sqrt{H_1}}; 0 \le u \le 1\right)\right] = E_\alpha\left[\frac{1(1-G_1 = V_1)}{1-G_1} F(\rho(u); 0 \le u \le 1)\right] \quad (64)$$

To obtain (62) from (64), observe that G_{H_1}/H_1 is the last zero before time 1 of $(\rho(uH_1)/\sqrt{H_1}; 0 \le u \le 1)$, and consequently

$$E_\alpha[F(\tilde\rho)] = E_\alpha\left[\frac{1(1-G_1 = V_1)}{1-G_1} F(\pi)\right] \quad (65)$$

Formula (62) now appears as a consequence of

$$E_\alpha\left[\frac{1(1-G_1 = V_1)}{1-G_1}\,\middle|\,\pi\right] = \frac{\gamma_\alpha}{(V_1(\pi))^\alpha} \quad (66)$$

To check (66), evaluate the left side of (66) as

$$E_\alpha\left[\frac{1\{(1-G_1)/G_1 > V_1(\pi)\}}{1-G_1}\,\middle|\,\pi\right] = h_\alpha(V_1(\pi))$$

where

$$h_\alpha(v) := E_\alpha\left[\frac{1}{1-G_1} 1\left(\frac{1-G_1}{G_1} > v\right)\right] = (\alpha\Gamma(\alpha)\Gamma(1-\alpha)v^\alpha)^{-1},$$

the last equality being a consequence of (61). □

Proof of (23). For $t > 0$ and $n = 1, 2, \ldots$ let $R_n(t) := V_{n+1}(t)/V_n(t)$. Since H_1 is admissible,

$$(R_1(H_1), R_2(H_1), \ldots) \stackrel{d}{=} (R_1(1), R_2(1), \ldots). \quad (67)$$

According to Proposition 8 of [17], the $R_n(1)$ are independent, and $R_n(1)$ has a beta$(n\alpha, 1)$ distribution. Now

$$R_1(D_{H_1}) = \frac{V_2(H_1)}{D_{H_1} - G_{H_1}} = R_1(H_1)(D_{H_1} - G_{H_1})^{-1} \quad (68)$$

and $R_m(D_{H_1}) = R_m(H_1)$ for $m \geq 2$. Since $D_{H_1} - G_{H_1}$ is independent of the sequence $(V_1(H_1), V_2(H_1), \ldots)$, for a generic non-negative product measurable f, we obtain

$$E_\alpha[\, f(V_1(D_{H_1}), V_2(D_{H_1}), \ldots)\,] = E_\alpha[\, \xi_\alpha(R_1(H_1))\, f(V_1(H_1), V_2(H_1), \ldots)\,] \tag{69}$$

and hence from (67)

$$E_\alpha\left[f\left(\frac{V_1(D_{H_1})}{D_{H_1}}, \frac{V_2(D_{H_1})}{D_{H_1}}, \ldots\right)\right] = E_\alpha[\, \xi_\alpha(V_2/V_1)\, f(V_1, V_2, \ldots)\,] \tag{70}$$

where

$$\xi_\alpha(x) := \frac{P_\alpha(R_1(D_{H_1}) \in dx)}{P_\alpha(R_1(1) \in dx)} = -\alpha \log x \tag{71}$$

The last equality follows by elementary computation from the fact that under P_α the distribution of $R_1(1)$ is beta$(\alpha, 1)$ while $P_\alpha(D_{H_1} - G_{H_1} > t) = t^{-\alpha}$ for $t > 1$. □

To conclude this section we note that there are analogs of the above formulae for H_n instead of H_1. For example, formula (22) is modified by replacing $S_1 V_1^{-\alpha}$ by $S_1(V_n^{-\alpha} - V_{n-1}^{-\alpha})$, which is also exponentially distributed [18, Prop. 10], and formula (62) is modified by replacing $V_1^{-\alpha}$ by $V_n^{-\alpha} - V_{n-1}^{-\alpha}$.

4 Generalized arc-sine laws.

In this section, we assume that $0 < \alpha < 1, 0 < p < 1$, and let $P_{\alpha,p}$ govern a real-valued process $(B_t, t \geq 0)$ with continuous paths, such that

(i) the zero set Z of B is the range of a stable (α) subordinator, and

(ii) given $|B|$, the signs of excursions of B away from zero are chosen independently of each other to be positive with probability p and negative with probability $q := 1 - p$.

For example, B could be any of the following:

- an ordinary Brownian motion ($\alpha = p = \frac{1}{2}$) [11]
- a *skew Brownian motion* ($\alpha = \frac{1}{2}, 0 < p < 1$) [21, 6, 2, 1]
- a *symmetrized Bessel process* of dimension $2 - 2\alpha$ [10]

- a *skew Bessel process of dimension* $2-2\alpha$ [2, 23]

For $t > 0$ let

$$A_t := \int_0^t 1(B_s > 0)\, ds \tag{72}$$

denote the time spent positive by B up to time t. See the papers cited above for background and motivation for the study of this process. For any random time T which is a measurable function of $|B|$,

$$A_T = \int_0^T 1(B_s > 0)\, ds = \sum_n \varepsilon_n(T) V_n(T) \tag{73}$$

where under $P_{\alpha,p}$ the $\varepsilon_n(T)$ are independent indicators of events with probability p, independent of the sequence of ranked lengths $(V_n(T), n = 1, 2, \cdots)$ of component intervals of $[0,T]\backslash Z$. Consequently, the $P_{\alpha,p}$ distribution of A_T/T is the same for such T that are admissible for the zero set of B, and this common distribution is the $P_{\alpha,p}$ distribution of $A := A_1$. This is Lamperti's [9] generalized arc-sine distribution on $[0,1]$, determined by its Stieltjes transform

$$E_{\alpha,p}\left[\frac{1}{\lambda + A}\right] = \frac{p(1+\lambda)^{\alpha-1} + q\lambda^{\alpha-1}}{p(1+\lambda)^{\alpha} + q\lambda^{\alpha}} \qquad (\lambda > 0) \tag{74}$$

Let $P^{\mathrm{br}}_{\alpha,p}$ denote the standard bridge law obtained by conditioning $P_{\alpha,p}$ on $(1 \in Z)$. If $P_{\alpha,p}$ governs B as a skew Bessel process, $P^{\mathrm{br}}_{\alpha,p}$ governs B as a skew Bessel bridge of length 1. According to formula (4.b') of [2],

$$E^{\mathrm{br}}_{\alpha,p}\left[\frac{1}{(1+\lambda A)^{\alpha}}\right] = \frac{1}{p(1+\lambda)^{\alpha} + q} \qquad (\lambda > 0) \tag{75}$$

Lamperti [9] inverted the Stieltjes transform (74) to obtain the corresponding density on $[0,1]$, which is reproduced in [15] and [23]. We do not know how to invert (75) to obtain such an explicit formula in the bridge case for general α with $0 < \alpha < 1$, but it is a famous result of Lévy [11] that for the standard Brownian bridge, with $\alpha = p = 1/2$, the distribution of A is simply uniform on $[0,1]$.

We note that the $P_{\alpha,p}$ distribution of A is uniquely determined by formula (75), since by differentiating k times we obtain for $k = 1, 2, \cdots$

$$E^{\mathrm{br}}_{\alpha,p}\left[\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)A^k}{(1+\lambda A)^{\alpha+k}}\right] = (-1)^k \frac{d^k}{d\lambda^k}\left(\frac{1}{p(1+\lambda)^{\alpha} + q}\right) \qquad (\lambda > 0) \tag{76}$$

so we recover the moments

$$E^{\mathrm{br}}_{\alpha,p}(A^k) = \frac{(-1)^k}{\alpha(\alpha+1)\cdots(\alpha+k-1)} \frac{d^k}{d\lambda^k}\left(\frac{1}{p(1+\lambda)^\alpha + q}\right)\Bigg|_{\lambda=0} \tag{77}$$

In particular, from (74) and (77), for all $0 < \alpha < 1$ and $0 < p < 1$, we obtain the means

$$E^{\mathrm{br}}_{\alpha,p}(A) = E_{\alpha,p}(A) = p \tag{78}$$

which is also obvious from (72) and $P_{\alpha,p}(B_t > 0) = P^{\mathrm{br}}_{\alpha,p}(B_t > 0) = p$ for all $0 < t < 1$, and the variances

$$Var^{\mathrm{br}}_{\alpha,p}(A) = \frac{(1-\alpha)pq}{1+\alpha} < (1-\alpha)pq = Var_{\alpha,p}(A) \tag{79}$$

The inequality between the variances can be understood intuitively as follows. Conditioning to return to zero at time 1 tends to make the intervals smaller and more evenly distributed in length. So there is less variability in the fraction of time spent positive. For fixed p, as α increases from $0+$ to $1-$, both variances decrease, from the variance pq of a Bernoulli(p) variable ϵ_p at $\alpha = 0+$, down to variance 0 at $\alpha = 1-$. Consequently, under either $P_{\alpha,p}$ or $P^{\mathrm{br}}_{\alpha,p}$

$$A \xrightarrow{d} \begin{cases} p & \text{as } \alpha \uparrow 1 \\ \epsilon_p & \text{as } \alpha \downarrow 0 \end{cases} \tag{80}$$

where $\xrightarrow{d}$ denotes convergence in distribution. This behaviour can also be understood from the representation (73) and the observation that under either $P_{\alpha,p}$ or $P^{\mathrm{br}}_{\alpha,p}$

$$V_1(1) \xrightarrow{d} \begin{cases} 0 & \text{as } \alpha \uparrow 1 \\ 1 & \text{as } \alpha \downarrow 0 \end{cases} \tag{81}$$

See [17] for details and further references concerning the exact distribution of $V_1(1)$ under $P_{\alpha,p}$ and $P^{\mathrm{br}}_{\alpha,p}$.

Let $G := G_1$ be the time of the last zero of B before time 1. To conclude this section, we record the following proposition which describes the $P_{\alpha,p}$ distribution of A_G by a surprisingly simple density relative to the $P_{\alpha,p}$ distribution of $A := A_1$ discussed above. Combined with Lamperti's formula for the density of A_1, this yields an explicit formula for the density of A_G relative to Lebesgue measure.

Proposition 15 *For all* $0 < \alpha < 1$ *and* $0 < p < 1$,

$$P_{\alpha,p}(A_G \in dx) = \frac{1-x}{1-p} P_{\alpha,p}(A_1 \in dx) \qquad (0 < x < 1) \qquad (82)$$

Proof. Write E for $E_{\alpha,p}$. Then for all Borel measurable $f : [0,1] \to [0,\infty)$

$$\begin{aligned}(1-p)E[f(A_G)] &= E\left[f(A_G)1_{(B_1 \leq 0)}\right] \\ &= E\left[f(A_1)1_{(B_1 \leq 0)}\right] \\ &= E\left[f(A_1)(1-A_1)\right]\end{aligned}$$

where the first equality is due to the independence of A_G and the event $(B_1 < 0)$, the second is obvious, and the third is deduced from the formula

$$P_{\alpha,p}(B_1 \leq 0 \,|\, A_1) = 1 - A_1 \qquad (83)$$

which, as noted in [15], is an easy consequence of the sampling property (9). □

As a consequence of (82), the moments of A_G can expressed simply in terms of those of $A := A_1$ which can be read from (74). Assume now for simplicity that B is a skew Bessel process under $P_{\alpha,p}$. As noted in [2], we can write

$$A_G = G A^{\mathrm{br}} \qquad (84)$$

where G has beta$(\alpha, 1-\alpha)$ distribution, and A^{br} is the fraction of time spent positive by the skew Bessel bridge of length 1 obtained by rescaling of B on the random interval $[0, G]$. So the $P_{\alpha,p}$ distribution of A^{br} is identical to the $P^{\mathrm{br}}_{\alpha,p}$ distribution of $A := A_1$ discussed before. In principle, (84) determines this distribution of A^{br} in terms of the distributions of G and A_G just described. This gives an alternative formula to (77) for computing moments of A^{br}, hence some tricky algebraic identities, but unfortunately does not seem to yield any more explicit description of the law of A^{br}.

References

[1] M. Barlow. Skew brownian motion and a one-dimensional differential equation. *Stochastics*, 25:1–2, 1988.

[2] M. Barlow, J. Pitman, and M. Yor. Une extension multidimensionnelle de la loi de l'arc sinus. In *Séminaire de Probabilités XXIII*, pages 294–314. Springer, 1989. Lecture Notes in Math. 1372.

[3] E. B. Dynkin. Some limit theorems for sums of independent random variables with infinite mathematical expectations. *IMS-AMS Selected Translations in Math. Stat. and Prob.*, 1:171–189, 1961.

[4] B. Fristedt and S. J. Taylor. Constructions of local time for a Markov process. *Z. Wahrsch. Verw. Gebiete*, 62:73 – 112, 1983.

[5] P. Greenwood and J. Pitman. Construction of local time and Poisson point processes from nested arrays. *Journal of the London Mathematical Society*, 22:182–192, 1980.

[6] J. M. Harrison and L. A. Shepp. On skew Brownian motion. *The Annals of Probability*, 9:309 – 313, 1981.

[7] J. F. C. Kingman. Random discrete distributions. *J. Roy. Statist. Soc. B*, 37:1–22, 1975.

[8] F.B. Knight. On the duration of the longest excursion. In E. Cinlar, K.L. Chung, and R.K. Getoor, editors, *Seminar on Stochastic Processes*, pages 117–148. Birkhäuser, 1985.

[9] J. Lamperti. An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.*, 88:380 – 387, 1958.

[10] J. Lamperti. An invariance principle in renewal theory. *Ann. Math. Stat.*, 33:685 – 696, 1962.

[11] P. Lévy. Sur certains processus stochastiques homogènes. *Compositio Math.*, 7:283–339, 1939.

[12] M. Perman. Order statistics for jumps of normalized subordinators. *Stoch. Proc. Appl.*, 46:267–281, 1993.

[13] M. Perman, J. Pitman, and M. Yor. Size-biased sampling of Poisson point processes and excursions. *Probability Theory and Related Fields*, 92:21–39, 1992.

[14] J. Pitman. Partition structures derived from Brownian motion and stable subordinators. Technical Report 346, Dept. Statistics, U.C. Berkeley, 1992. To appear in *Bernoulli*.

[15] J. Pitman and M. Yor. Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc. (3)*, 65:326–356, 1992.

[16] J. Pitman and M. Yor. Some conditional expectation given an average of a stationary or self-similar random process. Technical Report 438, Dept. Statistics, U.C. Berkeley, 1995. In preparation.

[17] J. Pitman and M. Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. Technical Report 433, Dept. Statistics, U.C. Berkeley, 1995. To appear in *The Annals of Probability*.

[18] J. Pitman and M. Yor. Random discrete distributions derived from self-similar random sets. *Electronic J. Probability*, 1:Paper 4, 1–28, 1996.

[19] C.L. Scheffer. The rank of the present excursion. *Stoch. Proc. Appl.*, 55:101–118, 1995.

[20] M. S. Taqqu. A bibliographical guide to self-similar processes and long-range dependence. In *Dependence in Probab. and Stat.: A Survey of Recent Results; Ernst Eberlein, Murad S. Taqqu (Ed.)*, pages 137–162. Birkhäuser (Basel, Boston), 1986.

[21] J. Walsh. A diffusion with a discontinuous local time. In *Temps Locaux*, volume 52-53 of *Astérisque*, pages 37–45. Soc. Math. de France, 1978.

[22] S. Watanabe. On time inversion of one-dimensional diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 31:115–124, 1975.

[23] S. Watanabe. Generalized arc-sine laws for one-dimensional diffusion processes and random walks. In *Proceedings in Symposia in Pure Mathematics*, volume 57, pages 157–172. A.M.S., 1995.

[24] J.G. Wendel. Zero-free intervals of semi-stable Markov processes. *Math. Scand.*, 14:21 – 34, 1964.

[25] M. Yor. *Some Aspects of Brownian Motion*. Lectures in Math., ETH Zürich. Birkhaüser, 1992. Part I: Some Special Functionals.

[26] M. Yor. Random Brownian scaling and some absolute continuity relationships. In E. Bolthausen, M. Dozzi, and F. Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications. Centro Stefano Franscini, Ascona, 1993*, pages 243–252. Birkhäuser, 1995.