DIRICHLET FORMS ON TOTALLY DISCONNECTED SPACES AND BIPARTITE MARKOV CHAINS

DAVID ALDOUS AND STEVEN N. EVANS

ABSTRACT. We use Dirichlet form methods to construct and analyse a general class of reversible Markov processes with totally disconnected state spaces. We study in detail the special case of bipartite Markov chains. The latter processes have a state space consisting of an "interior" with a countable number of isolated points and a, typically uncountable, "boundary". The equilibrium measure assigns all of its mass to the interior. When the chain is started at a state in the interior, it holds for an exponentially distributed amount of time and then jumps to the boundary. It then instantaneously re-enters the interior. There is a "local time on the boundary". That is, the set of times the process is on the boundary is uncountable and coincides with the points of increase of a continuous additive functional. Certain processes with values in the space of trees and the space of vertices of a fixed tree provide natural examples of bipartite chains. Moreover, time–changing a bipartite chain by its local time on the boundary leads to interesting processes, including particular Lévy processes on local fields (for example, the p-adic numbers) that have been considered elsewhere in the literature.

1. INTRODUCTION

In [4] weak convergence methods were used to construct a rooted tree–valued Markov process called there the *wild chain*. This process arises naturally as a limiting case of tree–valued Markov chains considered in [5, 3]. The wild chain is reversible (that is, symmetric) with equilibrium measure the distribution of the critical Poisson Galton–Watson branching process (we denote this probability measure on rooted trees by PGW(1)). When started in a state that is a finite tree, the wild chain holds for an exponentially distributed amount of time and then jumps to a state that is an infinite tree. Then, as must be the case given that the PGW(1) distribution assigns all of its mass to finite trees, the process instantaneously re-enters the set of finite trees. In other words, the sample–paths of the wild chain bounce backwards and forwards between the finite and infinite trees. The set of times when the state of the wild chain is an infinite tree has Lebesgue measure zero, but it is the uncountable set of points of increase of a continuous additive functional (so that it looks qualitatively like the zero set of a Brownian motion).

The aim of this paper is to use Dirichlet form methods to construct and study a general class of symmetric Markov processes on a generic totally disconnected state

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space. Specialising this construction leads to processes that we call *bipartite chains* which have features similar to those of the wild chain.

In general, we take the state space of the processes we construct to be a Lusin space E such that there exists a countable algebra $\mathcal{R}$ of simultaneously closed and open subsets of E that is a base for the topology of E. Note that E is indeed totally disconnected (see Theorem 33.B of [15]). Conversely, if E is any totally disconnected compact metric space, then there exists a collection $\mathcal{R}$ with the required properties (see Theorem 2.94 of [11]).

The following are two instances of such spaces. More examples, including an arbitrary local field and the compactification of an infinite tree, are described in Section 2.

Example 1.1. Let E be $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, the usual one–point compactification of the positive integers $\mathbb{N} := \{1, 2, \ldots\}$. Equip E with the usual total order and let $\mathcal{R}$ be the algebra generated by sets of the form $\{y : x \le y\}$, $x \in \mathbb{N}$.

Example 1.2. Let E be the collection $\mathbf{T}_{\le\infty}$ of rooted trees with every vertex having finite out–degree. Write $\mathbf{T}_{\le n}$ for the subset of $\mathbf{T}_{\le\infty}$ consisting of trees with height at most n. For $m > n$, there is a natural projection map from $\rho_{mn} : \mathbf{T}_{\le m} \to \mathbf{T}_{\le n}$ that throws away vertices of height greater than n and the edges leading to them. We can identify $\mathbf{T}_{\le\infty}$ with the projective limit of this projective system and give it the corresponding projective limit topology (each $\mathbf{T}_{\le n}$ is given the discrete topology), so that $\mathbf{T}_{\le\infty}$ is Polish. Equip $\mathbf{T}_{\le\infty}$ with the inclusion partial order (that is, $x \le y$ if x is a sub–tree of y). Let $\mathcal{R}$ be the algebra generated by sets of the form $\{y : x \le y\}$, $x \in \mathbf{T}_{<\infty} := \bigcup_n \mathbf{T}_{\le n}$. Equivalently, if $\rho_n : \mathbf{T}_{\le\infty} \to \mathbf{T}_{\le n}$ is the projection map that throws away vertices of height greater than n and the edges leading to them, then $\mathcal{R}$ is the collection of sets of the form $\rho_n^{-1}(B)$ for finite or co-finite $B \subseteq \mathbf{T}_{\le n}$, as n ranges over $\mathbb{N}$.

Our main existence result is the following, which we prove in Section 3. We refer the reader to [10] and [12] for background on Dirichlet forms and their associated Markov processes.

Notation 1.3. Denote by $\mathcal{C}$ the subalgebra of $C_b(E)$ (:= continuous bounded functions on E) generated by the indicator functions of sets in $\mathcal{R}$.

Theorem 1.4. *Consider two probability measures μ and ν on E and a non-negative Borel function κ on $E \times E$. Define a σ-finite measure Λ on $E \times E$ by $\Lambda(dx, dy) := \kappa(x, y)\mu(dx)\nu(dy)$. Suppose that the following hold:*

(a) the closed support of the measure μ is E;
(b) $\Lambda([(E\backslash R) \times R] \cup [R \times (E\backslash R)]) < \infty$ for all $R \in \mathcal{R}$;
(c) $\int \kappa(x, y)\, \mu(dx) = \infty$ for ν_s-a.e. y, where ν_s is the singular component in the Lebesgue decomposition of ν with respect to μ;
(d) there exists a sequence $(R_n)_{n=1}^\infty$ of sets in $\mathcal{R}$ such that $\bigcap_{m=n}^\infty R_m$ is compact for all n, $\sum_{n=1}^\infty \mu(E\backslash R_n) < \infty$, and

$$\sum_{n=1}^{\infty} \Lambda([(E\backslash R_n) \times R_n] \cup [R_n \times (E\backslash R_n)]) < \infty.$$

Then there is a recurrent μ-symmetric Hunt process $\mathbf{X} = (X_t, \mathbb{P}^x)$ on E whose Dirichlet form is the closure of the form $\mathcal{E}$ on $\mathcal{C}$ defined by

$$\mathcal{E}(f,g) = \iint (f(y) \Leftrightarrow f(x))(g(y) \Leftrightarrow g(x))\, \Lambda(dx, dy), \;\; f, g \in \mathcal{C}.$$

Our standing assumption throughout the paper is that the conditions of Theorem 1.4 hold.

In order to produce processes that are reminiscent of the wild chain, we need to assume a little more structure on E. Say that E is *bipartite* if there is a countable, dense subset $E^o \subseteq E$ such that each point of E^o is isolated. In particular, E^o is open. In Example 1.1 one can take $E^o = \mathbb{N}$. In Example 1.2 one can take $E^o = T_{<\infty}$. We will see more examples in Section 2. Put $E^* = E \backslash E^o$. Note that E^* is the boundary of the open set E^o.

Definition 1.5. We will call the process $\mathbf{X}$ described in Theorem 1.4 a *bipartite Markov chain* if the space E is bipartite and, in the notation of Theorem 1.4:

- μ is concentrated on E^o,
- ν is concentrated on E^*.

Remark 1.6. The measures μ and ν are mutually singular and $\nu_s = \nu$ in the notation of Theorem 1.4. The reference measure μ is invariant for $\mathbf{X}$, that is, $\mathbb{P}^\mu\{X_t \in \cdot\} = \mu$ for each $t \geq 0$. Thus, for any $x \in E^o$ we have $\mathbb{P}^x\{X_t \in E^o\} = 1$ for each $t \geq 0$, and so $\mathbf{X}$ is Markov chain on the countable set E^o in the same sense that the Feller–McKean chain is a Markov chain on the rationals.

We establish in Proposition 4.2 that the sample–paths of $\mathbf{X}$ bounce backwards and forwards between E^o and E^* in the same manner that the sample paths of the wild chain bounce backwards and forwards between the finite and infinite trees. Also, we show in Proposition 4.4 that under suitable conditions μ is the unique invariant distribution for $\mathbf{X}$ that assigns all of its mass to E^o, and, moreover, for any probability measure γ concentrated on E^o the law of X_t under $\mathbb{P}^\gamma$ converges in total variation to μ.

In Section 6 we prove that, in the general setting of Theorem 1.4, the measure ν is the Revuz measure of a positive continuous additive functional (PCAF). We can therefore time–change $\mathbf{X}$ using the inverse of this PCAF. When this procedure is applied to a bipartite chain, it produces a Markov process with state space that is a subset of E^*. In particular, we observe in Example 7.2 that instances of this time–change construction lead to "spherically symmetric" Lévy processes on local fields that have previously been considered in [7, 1, 2, 9].

A useful tool for proving the last fact is a result from Section 5. There we consider a certain type of equivalence relation on E with associated map π onto the corresponding quotient space. We give conditions on the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ that are sufficient for the process $\pi \circ \mathbf{X}$ to be a symmetric Hunt process.

Notation 1.7. Write $(\cdot,\cdot)_\mu$ for the $L^2(E,\mu)$ inner product and $(T_t)_{t\geq 0}$ for the semi-group on $L^2(E,\mu)$ associated with the form $(\mathcal{E}, D(\mathcal{E}))$.

2. More examples of state spaces

Example 2.1. Let E be the usual path–space of a discrete–time Markov chain with countable state–space S augmented by a distinguished cemetery state ∂ to form $S^\partial = S \cup \{\partial\}$. That is, E is the subset of the space of sequences $(S^\partial)^{\mathbb{N}_0}$

(where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$) consisting of sequences $(x_n)_{n=0}^{\infty}$ such that if $x_n = \partial$ for some n, then $x_m = \partial$ for all $m > n$. Give E the subspace topology inherited from the product topology on $(S^\partial)^{\mathbb{N}_0}$ (where each factor has the discrete topology), so that E is Polish. Given $x \in E$, write $\zeta(x) := \inf\{n : x_n = \partial\} \in \mathbb{N}_0 \cup \{\infty\}$ for the death–time of x. Define a partial order on E by declaring that $x \leq y$ if $\zeta(x) \leq \zeta(y)$ and $x_n = y_n$ for $0 \leq n < \zeta(x)$. (In particular, if x and y are such that $\zeta(x) = \zeta(y) = \infty$, then $x \leq y$ if and only if $x = y$.) Let $\mathcal{R}$ be the algebra generated by sets of the form $\{y : x \leq y\}$, $\zeta(x) < \infty$. When $\#S = k < \infty$, we can think of E as the regular k-ary rooted tree along with its set of ends. In particular, when $k = 1$ we recover Example 1.1. This example is bipartite with $E^o = \{x : \zeta(x) < \infty\}$,

Example 2.2. A *local field* $\mathbb{K}$ is a locally compact, non-discrete, totally disconnected, topological field. We refer the reader to [16] or [13] for a full discusion of these objects and for proofs of the facts outlined below. More extensive summaries and references to the literature on probability in a local field setting can be found in [6] and [8].

There is a real-valued mapping on $\mathbb{K}$ which we denote by $x \mapsto |x|$. This map, called the *valuation* takes the values $\{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q = p^c$ for some prime p and positive integer c and has the properties

$$|x| = 0 \Leftrightarrow x = 0$$
$$|xy| = |x||y|$$
$$|x + y| \leq |x| \vee |y|.$$

The mapping $(x, y) \mapsto |x \Leftrightarrow y|$ on $\mathbb{K} \times \mathbb{K}$ is a metric on $\mathbb{K}$ which gives the topology of K.

Put $\mathbb{D} = \{x : |x| \leq 1\}$. The set D is a ring (the so-called *ring of integers* of K). If we choose $\rho \in K$ so that $|\rho| = q^{-1}$, then

$$\rho^k\mathbb{D} = \{x : |x| \leq q^{-k}\} = \{x : |x| < q^{-(k-1)}\}.$$

Every ball is of the form $x + \rho^k\mathbb{D}$ for some $x \in \mathbb{K}$ and $k \in \mathbb{Z}$, and, in particular, all balls are both closed and open. For $\ell < k$ the additive quotient group $\rho^\ell\mathbb{D}/\rho^k\mathbb{D}$ has order $q^{k-\ell}$. Consequently, $\mathbb{D}$ is the union of q disjoint translates of $\rho\mathbb{D}$. Each of these components is, in turn, the union of q disjoint translates of $\rho^2\mathbb{D}$, and so on. We can thus think of the collection of balls contained in $\mathbb{D}$ as being arranged in an infinite rooted q-ary tree: the root is $\mathbb{D}$ itself, the nodes at level k are the balls of radius q^{-k} (= cosets of $\rho^k\mathbb{D}$), and the q "children" of such a ball are the q cosets of $\rho^{k+1}\mathbb{D}$ that it contains. We can uniquely associate each point in $\mathbb{D}$ with the sequence of balls that contain it, and so we can think of the points in $\mathbb{D}$ as the ends this tree. This tree picture alone does not capture all the algebraic structure of $\mathbb{D}$; the rings of integers for the p-adic numbers and the p-series field (that is, the field of formal Laurent series with coefficients drawn from the finite field with p elements) are both represented by a p-ary tree, even though the p-adic field has characteristic 0 whereas the p-series field has characteristic p. (As an aside, a locally compact, non-discrete, topological field that is not totally disconnected is necessarily either the real or the complex numbers. Every local field is either a finite algebraic extension of the p-adic number field for some prime p or a finite algebraic extension of the p-series field.)

We can take either $E = \mathbb{K}$ or $E = \mathbb{D}$, with $\mathcal{R}$ the algebra generated by the balls. The same comment applies to Banach spaces over local fields defined as in [13], and we leave the details to the reader.

Example 2.3. In the notation of Example 1.2, let $\mathbf{T}^*_\infty$ be the subset of $\mathbf{T}_{\le\infty}$ consisting of infinite trees through which there is a unique infinite path starting at the root, that is, trees with only one end. Put $\mathbf{T}^* = \mathbf{T}_{<\infty} \cup \mathbf{T}^*_\infty$. It is not hard to see that $E = \mathbf{T}^*$ satisfies our hypothesis, with $\mathcal{R}$ the trace on $\mathbf{T}^*$ of the algebra of subsets of $\mathbf{T}_{\le\infty}$ described in Example 1.2.

Example 2.4. Suppose that the pairs $(E_1, \mathcal{R}_1), \dots, (E_N, \mathcal{R}_N)$ each satisfy our hypotheses. Put $E := \prod_i E_i$, equip E with the product topology, and set $\mathcal{R}$ to be the algebra generated by subsets of E of the form $\prod_i R_i$ with $R_i \in \mathcal{R}_i$. If each of the factors E_i is bipartite with corresponding countable dense sets of isolated point E_i^o, then E is also bipartite with countable dense of isolated points $\prod_i E_i^o$. Similar observations holds for sums rather than products, and we leave the details to the reader.

3. Proof of Theorem 1.4

We first check that $\mathcal{E}$ is well–defined on $\mathcal{C}$. Any $f \in \mathcal{C}$ can be written $f = \sum_{i=1}^N a_i \mathbf{1}_{R_i}$ for suitable $R_i \in \mathcal{R}$ and constants a_i, and condition (b) is just the condition that $\mathcal{E}(\mathbf{1}_R, \mathbf{1}_R) < \infty$ for all $R \in \mathcal{R}$. It is clear that $\mathcal{E}$ is a symmetric, non-negative, bilinear form on $\mathcal{C}$.

We next check that $\mathcal{E}$ defined on $\mathcal{C}$ is closable (as a form on $L^2(E, \mu)$). Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{C}$ such that

$$\lim_{n\to\infty} (f_n, f_n)_\mu = 0 \tag{3.1}$$

and

$$\lim_{m,n\to\infty} \mathcal{E}(f_m \Leftrightarrow f_n, f_m \Leftrightarrow f_n) = 0. \tag{3.2}$$

We need to show that

$$\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0. \tag{3.3}$$

Put $\Lambda_s(dx, dy) = \kappa(x, y)\, \mu(dx)\, \nu_s(dy)$. For $M > 0$ put $\Lambda^M(dx, dy) = [\kappa(x, y) \wedge M]\, \mu(dx)\, \nu(dy)$ and $\Lambda_s^M(dx, dy) = [\kappa(x, y) \wedge M]\, \mu(dx)\, \nu_s(dy)$. From (3.1) we have

$$\lim_{m,n\to\infty} \iint (f_m(x) \Leftrightarrow f_n(x))^2\ \Lambda_s^M(dx, dy) = 0,\ \forall M > 0,$$

and from (3.2) we have

$$\lim_{m,n\to\infty} \iint (\{f_m(y) \Leftrightarrow f_n(y)\} \Leftrightarrow \{f_m(x) \Leftrightarrow f_n(x)\})^2\ \Lambda^M(dx, dy) = 0,\ \forall M > 0. \tag{3.4}$$

So, by Minkowski's inequality,

$$\lim_{m,n\to\infty} \iint (f_m(y) \Leftrightarrow f_n(y))^2\ \Lambda_s^M(dx, dy) = 0,\ \forall M > 0. \tag{3.5}$$

Therefore, by (3.1), (3.5) and (c), there exists a Borel function f and a sequence $(n_k)_{k=1}^\infty$ such that $\lim_{k\to\infty} f_{n_k} = 0$, μ-a.e. (and hence ν_a-a.e., where $\nu_a = \nu \Leftrightarrow \nu_s$ is the absolutely continuous component in the Lebesgue decomposition of ν with respect to μ), and $\lim_{k\to\infty} f_{n_k} = f$, ν_s-a.e.

Now, by Fatou, (3.2) and Minkowski's inequality,

$$\begin{aligned}\iint f^2(y)\,\Lambda_s(dx,dy) &= \iint \lim_{k\to\infty} (f_{n_k}(y) \Leftrightarrow f_{n_k}(x))^2\ \Lambda_s(dx,dy)\\ &\le \liminf_{k\to\infty} \iint (f_{n_k}(y) \Leftrightarrow f_{n_k}(x))^2\ \Lambda_s(dx,dy)\\ &< \infty,\end{aligned}$$

and so, by (c), $f = 0$, ν_s-a.e. Finally, by Fatou and (3.2),

$$\begin{aligned}&\lim_{m\to\infty} \iint (f_m(y) \Leftrightarrow f_m(x))^2\ \Lambda(dx,dy)\\ &= \lim_{m\to\infty} \iint \lim_{k\to\infty} (\{f_m(y) \Leftrightarrow f_{n_k}(y)\} \Leftrightarrow \{f_m(x) \Leftrightarrow f_{n_k}(x)\})^2\ \Lambda(dx,dy)\\ &\le \lim_{m\to\infty} \liminf_{k\to\infty} \iint (\{f_m(y) \Leftrightarrow f_{n_k}(y)\} \Leftrightarrow \{f_m(x) \Leftrightarrow f_{n_k}(x)\})^2\ \Lambda(dx,dy)\\ &= 0,\end{aligned}$$

as required.

Write $(\mathcal{E}, D(\mathcal{E}))$ for the closure of the form $(\mathcal{E}, \mathcal{C})$. To complete the proof that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, it only remains to show that this form is Markov. By Proposition I.4.10 of [12], this will be accomplished if we can show for any $f \in \mathcal{C}$ that

$$(f \vee 0) \wedge 1 \in \mathcal{C} \tag{3.6}$$

and

$$\mathcal{E}((f \vee 0) \wedge 1, (f \vee 0) \wedge 1) \le \mathcal{E}(f,f). \tag{3.7}$$

Considering claim (3.6), first observe that $f \in \mathcal{C}$ if and only if there exist pairwise disjoint $R_1, \ldots, R_N$ and constants $a_1, \ldots, a_N$ such that $f = \sum_i a_i \mathbf{1}_{R_i}$. Thus,

$$(f \wedge 0) \vee 1 = \sum_i ((a_i \vee 0) \wedge 1)\mathbf{1}_{R_i} \in \mathcal{C}.$$

The claim (3.7) is immediate from the definition of $\mathcal{E}$ on $\mathcal{C}$.

We will appeal to Theorem 7.3.1 of [10] to establish that $(\mathcal{E}, D(\mathcal{E}))$ is the Dirichlet form of a μ-symmetric Hunt process, $\mathbf{X}$. It is immediate that conditions (C.1)–(C.3) of that result hold. Namely,

- $\mathcal{C}$ is a countably generated subalgebra of $D(\mathcal{E}) \cap C_b(E)$,
- $\mathcal{C}$ is $\mathcal{E}_1$-dense in $D(\mathcal{E})$ (that is, given any $f \in D(\mathcal{E})$ there exists a sequence $(f_n)_{n=1}^\infty$ of elements of $\mathcal{C}$ such that $\lim_{n\to\infty} \mathcal{E}(f_n \Leftrightarrow f, f_n \Leftrightarrow f) + (f_n \Leftrightarrow f, f_n \Leftrightarrow f)_\mu = 0$),
- $\mathcal{C}$ separates points of E and, for any $x \in E$, there is a $f \in \mathcal{C}$ such that $f(x) \neq 0$.

We therefore need only check the tightness condition; that is, that there exist compact sets $K_1 \subseteq K_2 \subseteq \ldots$ such that $\lim_{n\to\infty} \mathrm{Cap}(E \backslash K_n) = 0$. Take $K_n =$

$\bigcap_{m=n}^{\infty} R_m$. Then

$$\begin{aligned}\operatorname{Cap}(E \backslash K_n) &\leq \sum_{m=n}^{\infty} \operatorname{Cap}(E \backslash R_m) \\ &\leq \sum_{m=n}^{\infty} \left(\mathcal{E}(\mathbf{1}_{E\backslash R_m}, \mathbf{1}_{E\backslash R_m}) + (\mathbf{1}_{E\backslash R_m}, \mathbf{1}_{E\backslash R_m})_\mu\right) \\ &= \sum_{m=n}^{\infty} \left(\Lambda([(E\backslash R_m) \times R_m] \cup [R_m \times (E\backslash R_m)]) + \mu(E\backslash R_m)\right).\end{aligned}$$

The rightmost sum is finite by (d), and so we certainly have $\lim_{n\to\infty} \operatorname{Cap}(E\backslash K_n) = 0$.

Finally, because constants belong to $D(\mathcal{E})$, it follows from Theorem 1.6.3 of [10] that $\mathbf{X}$ is recurrent.

Remark 3.1. (i) Existing results on closability of "jump" forms in the literature (for example, Example 1.2.4 of [10]) appear to involve an absolute continuity condition that corresponds in our case to $\nu_s = 0$.

(ii) Suppose that $\mathcal{S} \subseteq \mathcal{R}$ generates $\mathcal{R}$, then it suffices to check condition (b) just for $R \in \mathcal{S}$, as the following argument shows. We remarked in the proof that condition (b) was just the statement that $\mathcal{E}(\mathbf{1}_R, \mathbf{1}_R) < \infty$ for all $R \in \mathcal{R}$. Note that $\mathbf{1}_R$ for $R \in \mathcal{R}$ is a finite linear combination of functions of the form $f = \prod_{i=1}^N \mathbf{1}_{S_i}$ for $S_1, \ldots, S_N \in \mathcal{S}$, and so it suffices to show that $\mathcal{E}(f,f) < \infty$ for such f. Observe that if $a_1, \ldots, a_N \in \mathbb{R}$ and $b_1, \ldots, b_N \in \mathbb{R}$ satisfy $|a_i| \leq 1$ and $|b_i| \leq 1$ for $1 \leq i \leq N$, then

$$\left|\prod_{i=1}^N a_i \Leftrightarrow \prod_{i=1}^N b_i\right| = \left|\sum_{i=1}^N \left(\prod_{j=1}^{i-1} a_j\right)(a_i \Leftrightarrow b_i)\left(\prod_{k=i+1}^N b_k\right)\right| \leq \sum_{i=1}^N |a_i \Leftrightarrow b_i|.$$

Therefore,

$$\begin{aligned}(f(y) \Leftrightarrow f(x))^2 &= |f(y) \Leftrightarrow f(x)| \\ &\leq \sum_{i=1}^N \left(\mathbf{1}_{(E\backslash S_i)\times S_i}(x,y) + \mathbf{1}_{S_i \times (E\backslash S_i)}(x,y)\right),\end{aligned}$$

and applying the assumption that (b) holds for all $R \in \mathcal{S}$ gives the result.

(iii) We emphasise that the elements of $D(\mathcal{E})$ are elements of $L^2(E,\mu)$ and are thus equivalence classes of functions. It is clear from the above proof that if $f, g \in D(\mathcal{E})$, then there are representatives $\hat{f}$ and $\hat{g}$ of the $L^2(E,\mu)$ equivalence classes of f and g such that

$$\mathcal{E}(f,g) = \iint (\hat{f}(y) \Leftrightarrow \hat{f}(x))(\hat{g}(y) \Leftrightarrow \hat{g}(x))\,\Lambda(dx,dy).$$

Some care must be exercised here: it is clear that if $\nu_s \neq 0$, then we cannot substitute an arbitrary choice of representatives into the right–hand side to compute $\mathcal{E}(f,g)$.

(iv) The above proof appealed to Theorem 7.3.1 of [10], which in turn relies on the regular representation results of Appendix A.4 of [10] to reduce to a locally compact setting. Therefore, although our state–space E is, in general, not locally compact, much of the theory developed in [10] for the locally compact setting still applies and we will use it without further comment.

We present several examples of set-ups satisfying the conditions of the Theorem 1.4 at the end of Section 4.

4. Bipartite chains

Assume for this section that $\mathbf{X}$ is a bipartite chain.

Notation 4.1. For a Borel set $B \subseteq E$, put $\sigma_B = \inf\{t > 0 : X_t \in B\}$ and $\tau_B = \inf\{t > 0 : X_t \notin B\}$.

Proposition 4.2. *(i) Consider $x \in E^o$. If $\int \kappa(x,z)\,\nu(dz) = 0$, then $\mathbb{P}^x\{\tau_{\{x\}} < \infty\} = 0$. Otherwise,*

$$\mathbb{P}^x\{\tau_{\{x\}} > t,\, X_{\tau_{\{x\}}} \in dy\} = \exp\left(\Leftrightarrow t \int \kappa(x,z)\,\nu(dz)\right) \frac{\kappa(x,y)\nu(dy)}{\int \kappa(x,z)\,\nu(dz)};$$

and, in particular, $\mathbb{P}^x\{X_{\tau_{\{x\}}} \in E^*\} = 1$.

(ii) For q.e. $x \in E^*$, $\mathbb{P}^x\{X_t \in E^o\} = 1$ for Lebesgue almost all $t \geq 0$. In particular, $\mathbb{P}^x\{\sigma_{E^o} = 0\} = 1$ for q.e. $x \in E^*$.

Proof. (i) Because each $x \in E^o$ is isolated, it follows from standard considerations that $\mathbb{P}\{\tau_{\{x\}} > t\} = \exp(\Leftrightarrow \alpha t)$, where

$$\begin{aligned}\mu(\{x\})\alpha &= \Leftrightarrow \lim_{t \downarrow 0} \left(\frac{1}{t}(T_t \Leftrightarrow I)\mathbf{1}_x, \mathbf{1}_x\right)_\mu \\ &= \mathcal{E}(\mathbf{1}_x, \mathbf{1}_x) = \mu(\{x\}) \int \kappa(x,z)\,\nu(dz).\end{aligned}$$

Observe for $f, g \in \mathcal{C}$ that $\mathcal{E}(f,g) = \iint (f(y) \Leftrightarrow f(x))(g(y) \Leftrightarrow g(x))\,J(dx,dy)$, where $J(dx,dy) = (1/2)[\Lambda(dx,dy) + \Lambda(dy,dx)]$ is the symmetrisation of Λ. Note that J is a symmetric measure that assigns no mass to the diagonal of $E \times E$. This representation of $\mathcal{E}$ is the one familiar from the Buerling–Deny formula. The result now follows from Lemma 4.5.5 of [10].

(ii) This is immediate from the Markov property, Fubini and the observation $\mathbb{P}^\mu\{X_t \notin E^o\} = \mu(E^*) = 0$ for all $t \geq 0$. □

Definition 4.3. Define a subprobability kernel ξ on E by $\xi(x,B) = \mu \otimes \nu(\{(x',y) : \kappa(x,y) > 0,\, \kappa(x',y) > 0,\, x' \in B\})$. Note that $\xi(x,\cdot) \leq \mu$. Say that $\mathbf{X}$ is *graphically irreducible* if there exists $x_0 \in E^o$ such that for all $x \in E^o$ there exists $n \in \mathbb{N}$ for which $\xi^n(x_0, \{x\}) > 0$.

Recall that a measure η is *invariant* for $\mathbf{X}$ if $\mathbb{P}^\eta\{X_t \in \cdot\} = \eta$ for all $t \geq 0$.

Proposition 4.4. *Suppose that $\mathbf{X}$ is graphically irredudicible. Then μ is the unique invariant probability measure for $\mathbf{X}$ such that $\mu(E^o) = 1$. If γ is any other probability measure such that $\gamma(E^o) = 1$, then*

$$\lim_{t \to \infty} \sup_B |\mathbb{P}^\gamma\{X_t \in B\} \Leftrightarrow \mu(B)| = 0.$$

Proof. By standard coupling arguments, both claims will hold if we can show

$$\mathbb{P}^x\{\sigma_{\{y\}} < \infty\} = 1, \text{ for all } x, y \in E^o. \tag{4.1}$$

For (4.1) it suffices by Theorem 4.6.6 of [10] to check that the recurrent form $\mathcal{E}$ is irreducible in the sense of Section 1.6 of [10]. Furthermore, applying Theorem

1.6.1 of [10] (and the fact that $1 \in D(\mathcal{E})$ with $\mathcal{E}(1,1)=0$), it is certainly enough to establish that if B is any Borel set with $\mathbf{1}_B \in D(\mathcal{E})$ and

$$0 = \mathcal{E}(\mathbf{1}_B, \mathbf{1}_B) + \mathcal{E}(\mathbf{1}_{E\setminus B}, \mathbf{1}_{E\setminus B}) = 2\mathcal{E}(\mathbf{1}_B, \mathbf{1}_B), \tag{4.2}$$

then $\mu(B)$ is either 0 or 1.

Suppose that (4.2) holds. By Remark 3.1(iii), there is a Borel function $\hat{f}$ with $\hat{f} = \mathbf{1}_B$, μ-a.e., such that

$$\begin{aligned} 0 &= \mathcal{E}(\mathbf{1}_B, \mathbf{1}_B) \\ &= \iint \left(\hat{f}(y) \Leftrightarrow \hat{f}(x)\right)^2 \Lambda(dx, dy) \\ &= \iint \left(\hat{f}(y) \Leftrightarrow \mathbf{1}_B(x)\right)^2 \Lambda(dx, dy). \end{aligned} \tag{4.3}$$

Suppose first that $x_0 \in B$, where x_0 is as in Definition 4.3. From (4.3),

$$\int \left(\hat{f}(y) \Leftrightarrow 1\right)^2 \kappa(x_0, y)\, \nu(dy) = 0,$$

and so $\nu(\{y : \hat{f} \neq 1, \kappa(x_0, y) > 0\}) = 0$. Therefore, again from (4.3), $\xi(x_0, \{x : \mathbf{1}_B(x) \neq 1\}) = 0$. That is, if $\xi(x_0, \{x\}) > 0$, then $x \in B$. Continuing in this way, we get that if $x \in E^o$ is such that $\xi^n(x_0, \{x\}) > 0$ for some n, then $x \in B$. Thus $E^o \subseteq B$ and $\mu(B) = 1$. A similar argument shows that if $x_0 \notin B$, then $\mu(B) = 0$. □

Example 4.5. Suppose that we are in the setting of Example 1.1 with $E^o = \mathbb{N}$. Let μ be an arbitrary fully supported probability measure on $\mathbb{N}$ and put $\nu = \delta_\infty$. In order that the conditions of Theorem 1.4 hold we only need κ to satisfy $\sum_{x\in\mathbb{N}} \kappa(x,\infty)\mu(\{x\}) = \infty$. The conditions of Proposition 4.4 will hold if and only if $\kappa(x,\infty) > 0$ for all $x \in \mathbb{N}$.

Example 4.6. We recall the Dirichlet form for the wild chain described in [4]. There $E = \mathbf{T}^*$ from Example 2.3, μ is the PGW(1) distribution and ν is the distribution of a PGW(1) tree "conditioned to be infinite". A more concrete description of ν is the following. Each $y \in \mathbf{T}^*_\infty$ has a unique path $(u_0, u_1, u_2, \ldots)$ starting at the root. There is a bijection between $\mathbf{T}^*_\infty$ and $\mathbf{T}_{<\infty} \times \mathbf{T}_{<\infty} \times \ldots$ that is given by identifying $y \in \mathbf{T}^*_\infty$ with the sequence of finite trees $(y_0, y_1, y_2, \ldots)$, where y_i is the tree rooted at u_i in the forest obtained by deleting the edges of the path $(u_0, u_1, u_2, \ldots)$. The probability measure ν on $\mathbf{T}^*_\infty$ is the push–forward by this bijection of the probability measure $\mu \times \mu \times \ldots$ on $\mathbf{T}_{<\infty} \times \mathbf{T}_{<\infty} \times \ldots$

Rather than describe $\kappa(x,y)$ explicitly, it is more convenient (and equally satisfactory for our purposes) to describe the measures $q^\uparrow(x, dy) := \kappa(x,y)\nu(dy)$ for each y and $q^\downarrow(y, dx) := \kappa(x,y)\mu(dx)$ for each x. Given $x \in \mathbf{T}_{<\infty}$, $y \in \mathbf{T}^*_\infty$, and a vertex u of x, let $(x/u/y) \in \mathbf{T}^*_\infty$ denote the tree rooted at the root of x that is obtained by inserting a new edge from u to the root of y. Then

$$q^\uparrow(x, f) := \sum_{u \in x} \int f((x/u/y))\, \nu(dy) \tag{4.4}$$

for f a non-negative Borel function on $\mathbf{T}^*$.

For $y \in \mathbf{T}^*_\infty$ with infinite path from the root $(u_0, u_1, u_2, \ldots)$ and $i \in \mathbb{N}_0$, removing the edge (u_i, u_{i+1}) produces two trees, one finite rooted at u_0 and one infinite rooted at u_{i+1}. Let $k_i(y) \in \mathbf{T}_{<\infty}$ denote the finite tree. Then (4.4) is equivalent to

$$q^{\downarrow}(y,f) = \sum_{i=0}^{\infty} f(k_i(y)) \tag{4.5}$$

for f a non-negative Borel function on $\mathbf{T}^*$.

Let us now check the conditions of Theorem 1.4. Condition (a) is obvious. Turning to condition (b), recall that any $R \in \mathcal{R}$ is of the form $\{x : \rho_n(x) \in B\}$ for some $n \in \mathbb{N}$ and finite or co-finite $B \subseteq \mathbf{T}_{\le n}$. Note that $[(\mathbf{T}^* \backslash R) \times R] \cup [R \times (\mathbf{T}^* \backslash R)] \subseteq \{(x,y) : \rho_n(x) \neq \rho_n(y)\}$. Moreover, if $y \in \mathbf{T}^*_\infty$ is of the form $(x/u/y')$ for some $u \in x$ and $y' \in \mathbf{T}^*_\infty$, then $\rho_n(x) \neq \rho_n(y)$ if and only if u has height less than n. Therefore, by (4.4),

$$\Lambda([(\mathbf{T}^* \backslash R) \times R] \cup [R \times (\mathbf{T}^* \backslash R)]) \le \int \#(\rho_{n-1}(x))\, \mu(dx) = n,$$

where we recall that the expected size of the k^{th} generation in a critical Galton–Watson branching process is 1.

It is immediate from (4.5) that

$$\int \kappa(x,y)\, \mu(dx) = q^{\downarrow}(y,1) = \infty$$

for $\nu = \nu_s$ almost every y, and so condition (c) holds.

Finally, consider condition (d). Put $S_{n,c} := \{x : \#(\rho_n(x)) \le c\}$. We will take $R_n = S_{n,c_n}$ for some sequence of constants $(c_n)_{n=1}^\infty$. Note that $\bigcap_{m=n}^\infty S_{m,c_m}$ is compact for all n, whatever the choice of $(c_n)_{n=1}^\infty$. By choosing c_n large enough, we can certainly make $\mu(\mathbf{T}^* \backslash S_{n,c_n}) \le 2^{-n}$. From the argument for part (b) we know that $[(\mathbf{T}^* \backslash S_{n,c}) \times S_{n,c}] \cup [S_{n,c} \times (\mathbf{T}^* \backslash S_{n,c})] = S_{n,c} \times (\mathbf{T}^* \backslash S_{n,c})$ is contained in the set $\{(x,y) : \rho_n(x) \neq \rho_n(y)\}$, which has finite Λ measure. Of course, $\lim_{c\to\infty} \mathbf{T}^* \backslash S_{n,c} = \emptyset$. Therefore, by dominated convergence, $\lim_{c\to\infty} \Lambda([(\mathbf{T}^* \backslash S_{n,c}) \times S_{n,c}] \cup [S_{n,c} \times (\mathbf{T}^* \backslash S_{n,c})]) = 0$, and by choosing c_n large enough we can make $\Lambda([(\mathbf{T}^* \backslash S_{n,c_n}) \times S_{n,c_n}] \cup [S_{n,c_n} \times (\mathbf{T}^* \backslash S_{n,c_n})]) \le 2^{-n}$.

It is obvious that the extra bipartite chain conditions hold with $E^o = \mathbf{T}_{<\infty}$. The condition of Proposition 4.4 also holds. More specifically, we can take x_0 in Definition 4.3 to be the trivial tree consisting of only a root. By (4.4) and (4.5), the measure $\xi^n(x_0, \cdot)$ assigns positive mass to every tree $x \in \mathbf{T}_{<\infty}$ with at most n children in the first generation (that is, $x \in \mathbf{T}_{<\infty}$ such that $\#(\rho_1(x)) \le n+1$), and so $\mathbf{X}$ is indeed graphically irreducible.

Example 4.7. Suppose that we are in the setting of Example 2.1 with $\#S < \infty$ (so that E is compact) and E^o the set $\{x : \zeta(x) < \infty\}$, as above. Note that $E^* = S^{\mathbb{N}_0}$. Fix a probability measure P on S with full support, an $S \times S$ stochastic matrix Q with positive entries and a probability measure R on $\mathbb{N}_0$. Define a probability measure μ on E^o by $\mu(\{x : \zeta(x) = n, x_0 = s_0, \ldots, x_{n-1} = s_{n-1}\}) = R(n)P(s_0)Q(s_0,s_1)\ldots Q(s_{n-2},s_{n-1})$. In other words, μ is the law of a Markov chain with initial distribution P and transition matrix Q killed at an independent time with distribution R. Define ν on E^* by $\nu(\{s_0\} \times \cdots \times \{s_n\} \times S \times S \times \ldots) = P(s_0)Q(s_0,s_1)\ldots Q(s_{n-1},s_n)$. Thus ν is the law of the unkilled chain with initial distribution P and transition matrix Q. Define $\kappa(x,y)$ for $x \in E^o$ and $y \in E^*$ by $\kappa(x,y) = K(\zeta(x))\mathbf{1}_{x \le y}$ for some sequence of non-negative constants $K(n)$, $n \in \mathbb{N}_0$.

In order that the conditions of Theorem 1.4 hold, we only need K to satisfy $\sum_{x \le y} K(\zeta(x))\mu(\{x\}) = \infty$ for ν-a.e. $y \in E^*$. For example, if $q_* = \min_{s,s'} Q(s,s')$, then it suffices that $\sum_{n=0}^{\infty} K(n)R(n)q_*^n = \infty$. In particular, if ν is the law of a sequence of i.i.d. uniform draws from S (so that $P(s) = S(s, s') = (\#S)^{-1}$ for all $s, s' \in S$), then we require $\sum_{n=0}^{\infty} K(n)R(n)(\#S)^{-n} = \infty$.

In general, $\mathbf{X}$ will be graphically irreducible with $x_0 = (\partial, \partial, \dots)$ (and hence the condition of Proposition 4.4 holds) if $K(n) > 0$ for all $n \in \mathbb{N}_0$.

5. Quotient processes

Return to the general set-up of Theorem 1.4. Suppose that $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$ and write $\mathcal{C}'$ for the subalgebra of $\mathcal{C}$ generated by the indicator functions of sets in $\mathcal{R}'$. We can define an equivalence relation on E by declaring that x and y are equivalent if $f(x) = f(y)$ for all $f \in \mathcal{C}'$. Let $\bar{E}$ denote the corresponding quotient space equipped with the quotient topology and denote by $\pi : E \to \bar{E}$ the quotient map. It is not hard to check that $\bar{E}$ is a Lusin space and that the algebra $\bar{\mathcal{R}} := \{\pi R : R \in \mathcal{R}'\}$ consists of simultaneously closed and open sets and is a base for the topology of $\bar{E}$. Write $\bar{\mathcal{C}}$ for the algebra generated by the indicator functions of sets in $\bar{\mathcal{R}}$. Note that $\mathcal{C}' = \{\bar{f} \circ \pi : \bar{f} \in \bar{\mathcal{C}}\}$.

Proposition 5.1. *Suppose that the following hold:*

(a) $\mu = \nu$;
(b) there exists a Borel function $\bar{\kappa} : \bar{E} \times \bar{E} \to \mathbb{R}_+$ *such that* $\kappa(x, y) = \bar{\kappa}(\pi x, \pi y)$ *for* $\pi x \neq \pi y$;
(c) $\bar{E}$ *is compact;*
(d) $\mu_{\mathcal{R}'}[f] := \mu[f|\sigma(\mathcal{R}')] = \mu[f|\sigma(\pi)]$ *has a version in* $\mathcal{C}'$ *for all* $f \in \mathcal{C}$.

Then the hypotheses of Theorem 1.4 hold with E, $\mathcal{R}$, $\mathcal{C}$, μ, ν, κ *replaced by* $\bar{E}$, $\bar{\mathcal{R}}$, $\bar{\mathcal{C}}$, $\bar{\mu}$, $\bar{\nu}$, $\bar{\kappa}$, *where* $\bar{\mu} = \bar{\nu}$ *is the push–forward of* $\mu = \nu$ *by* π. *Moreover, if* $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ *denotes the resulting Dirichlet form, then* $\pi \circ \mathbf{X}$ *is a* $\bar{\mu}$*-symmetric Hunt process with Dirichlet form* $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$.

Proof. It is clear that the hypotheses of Theorem 1.4 hold with $E, \mathcal{R}, \mathcal{C}, \mu, \nu, \kappa$ replaced by $\bar{E}, \bar{\mathcal{R}}, \bar{\mathcal{C}}, \bar{\mu}, \bar{\nu}, \bar{\kappa}$.

Let $(\bar{T}_t)_{t\geq 0}$ denote the semigroup on $L^2(\bar{E}, \bar{\mu})$ corresponding to $\bar{\mathcal{E}}$. The proof $\pi \circ \mathbf{X}$ is a $\bar{\mu}$-symmetric Hunt process with Dirichlet form $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ will be fairly straightforward once we establish that $T_t(\bar{f} \circ \pi) = (\bar{T}_t \bar{f}) \circ \pi$ for all $t \geq 0$ and $\bar{f} \in L^2(\bar{E}, \bar{\mu})$ (see Theorem 13.5 of [14] for a proof that this suffices for $\pi \circ \mathbf{X}$ to be a Hunt process – the proof that $\pi \circ \mathbf{X}$ is $\bar{\mu}$-symmetric and the identification of the associated Dirichlet form are then easy). Equivalently, writing $(G_\alpha)_{\alpha>0}$ and $(\bar{G}_\alpha)_{\alpha>0}$ for the resolvents corresponding to $(T_t)_{t\geq 0}$ and $(\bar{T}_t)_{t\geq 0}$, we need to establish that $G_\alpha(\bar{f} \circ \pi) = (\bar{G}_\alpha \bar{f}) \circ \pi$ for all $\alpha > 0$ and $\bar{f} \in L^2(\bar{E}, \bar{\mu})$. This is further equivalent to establishing that $(\bar{G}_\alpha \bar{f}) \circ \pi \in D(\mathcal{E})$ and $\mathcal{E}((\bar{G}_\alpha \bar{f}) \circ \pi, g) + \alpha((\bar{G}_\alpha \bar{f}) \circ \pi, g)_\mu = (\bar{f} \circ \pi, g)_\mu$ for all $g \in \mathcal{C}$ (see Equation (1.3.7) of [10]).

Fix $\bar{f} \in L^2(\bar{E}, \bar{\mu})$ and $g \in \mathcal{C}$. By assumption, $\mu_{\mathcal{R}'}[g] = \bar{g} \circ \pi$ for some $\bar{g} \in \bar{\mathcal{C}}$. Also, it is fairly immediate from the definition of $\bar{E}$ that $\bar{h} \in D(\bar{\mathcal{E}})$ if and only if

$\bar h \circ \pi \in D(\mathcal{E})$, and that $\bar{\mathcal{E}}(\bar h, \bar h) = \mathcal{E}(\bar h \circ \pi, \bar h \circ \pi)$. Hence, by Remark 3.1(iii),

$$\begin{aligned}
\mathcal{E}(\bar h \circ \pi, g) &= \iint \left(\bar h \circ \pi(y) \Leftrightarrow \bar h \circ \pi(x)\right) (g(y) \Leftrightarrow g(x))\, \Lambda(dx, dy) \\
&= \iint_{\{(x,y):\pi x \neq \pi y\}} \left(\bar h \circ \pi(y) \Leftrightarrow \bar h \circ \pi(x)\right) (g(y) \Leftrightarrow g(x))\, \Lambda(dx, dy) \\
&= \iint \left(\bar h \circ \pi(y) \Leftrightarrow \bar h \circ \pi(x)\right) (g(y) \Leftrightarrow g(x))\, \bar\kappa(\pi x, \pi y)\, \mu(dx)\, \mu(dy) \\
&= \iint \left(\bar h \circ \pi(y) \Leftrightarrow \bar h \circ \pi(x)\right) (\mu_{\mathcal{R}'}[g](y) \Leftrightarrow \mu_{\mathcal{R}'}[g](x))\, \bar\kappa(\pi x, \pi y)\, \mu(dx)\, \mu(dy) \\
&= \iint \left(\bar h \circ \pi(y) \Leftrightarrow \bar h \circ \pi(x)\right) (\bar g \circ \pi(y) \Leftrightarrow \bar g \circ \pi(x))\, \bar\kappa(\pi x, \pi y)\, \mu(dx)\, \mu(dy) \\
&= \iint \left(\bar h(w) \Leftrightarrow \bar h(v)\right) (\bar g(w) \Leftrightarrow \bar g(v))\, \bar\kappa(v, w)\, \bar\mu(dv)\, \bar\mu(dw) \\
&= \bar{\mathcal{E}}(\bar h, \bar g).
\end{aligned}$$

Of course,

$$(\bar h \circ \pi, g)_\mu = (\bar h \circ \pi, \bar g \circ \pi)_\mu = (\bar h, \bar g)_{\bar\mu}.$$

Therefore,

$$\begin{aligned}
&\mathcal{E}((\bar G_\alpha \bar f) \circ \pi, g) + \alpha((\bar G_\alpha \bar f) \circ \pi, g)_\mu = \bar{\mathcal{E}}(\bar G_\alpha \bar f, \bar g) + \alpha(\bar G_\alpha \bar f, \bar g)_{\bar\mu} \\
&\quad = (\bar f, \bar g)_{\bar\mu} = (\bar f \circ \pi, \bar g \circ \pi)_\mu = (\bar f \circ \pi, g)_\mu,
\end{aligned}$$

as required. □

We will see an application of Proposition 5.1 at the end of Section 7.

6. Additive functionals

We are still in the general setting of Theorem 1.4.

Proposition 6.1. *The probability measure ν assigns no mass to sets of zero capacity, and there is a positive continuous additive functional $(A_t)_{t\geq 0}$ with Revuz measure ν.*

Proof. The reference measure μ assigns no mass to sets of zero capacity, so it suffices to show that ν_s assigns no mass to sets of zero capacity. For $M > 0$ put $G_M := \{y : \int [\kappa(x,y) \wedge M]\, \mu(dx) \geq 1\}$ and define a subprobability measure ν_s^M by $\nu_s^M := \nu_s(\cdot \cap G_M)$. By (c) of Theorem 1.4, $\nu_s(E \backslash \bigcup_M G_M) = 0$, and so it suffices to show for each M that ν_s^M assigns no mass to sets of zero capacity.

Observe for $f \in \mathcal{C}$ that

$$\begin{aligned}
\left(\int |f(y)|\, \nu_s^M(dy)\right)^2 &\leq \int f^2(y)\, \nu_s^M(dy) \leq \iint f^2(y)\, \Lambda^M(dx, dy) \\
&\leq 2\left(\iint (f(y) \Leftrightarrow f(x))^2\, \Lambda^M(dx, dy) + \iint f^2(x)\, \Lambda^M(dx, dy)\right) \\
&\leq 2(1 \vee M)\left(\mathcal{E}(f,f) + (f,f)_\mu\right).
\end{aligned}$$

The development leading to Lemma 2.2.3 of [10] can now be followed to show that for all Borel sets B we have $\nu_s^M(B) \leq C_M \mathrm{Cap}(B)^{1/2}$ for a suitable constant C_M (the argument in [10] is in a locally compact setting, but it carries over without difficulty to our context).

The existence and uniqueness of $(A_t)_{t \geq 0}$ follows from Theorem 5.1.4 of [10]. □

Remark 6.2. In the bipartite chain case, the distribution under $\mathbb{P}^\mu$ of X_ζ, where $\zeta := \tau_{\{X_0\}}$, is mutually absolutely continuous with respect to ν, and Proposition 6.1 is obvious.

7. Bipartite chains on the boundary

Return to the bipartite chain setting. Following the construction in Section 6.2 of [10], let $\mathbf{Y}$ denote the process $\mathbf{X}$ time–changed according to the positive continuous additive functional A. That is, $Y_t = X_{\gamma_t}$ where $\gamma_t = \inf\{s > 0 : A_s > t\}$. Write $\tilde{E}$ for the support of A. We have $\tilde{E} \subseteq \breve{E} := \operatorname{supp} \nu \subseteq E^*$ and $\nu(\breve{E} \backslash \tilde{E}) = 0$.

Let $\breve{\mathcal{R}} = \{R \cap \breve{E} : R \in \mathcal{R}\}$ and put $\breve{\mathcal{C}} = \{f_{|\breve{E}} : f \in \mathcal{C}\}$. Note that $\breve{\mathcal{C}}$ is also the algebra generated by $\breve{\mathcal{R}}$.

Theorem 7.1. *The process $\mathbf{Y}$ is a recurrent ν-symmetric Hunt process with state–space $\breve{E}$ and Dirichlet form given by the closure of the form $\breve{\mathcal{E}}$ on $\breve{\mathcal{C}}$ defined by*

$$\breve{\mathcal{E}}(f, g) = \iint (f(y) \Leftrightarrow f(z))\,(g(y) \Leftrightarrow g(z))\,\breve{\kappa}(y, z)\,\nu(dy)\,\nu(dz), \;\; f, g \in \breve{\mathcal{C}},$$

where

$$\breve{\kappa}(y, z) = \int \kappa(x, y) \frac{\kappa(x, z)}{\int \kappa(x, w)\,\nu(dw)}\,\mu(dx)$$

(with the convention $0/0 = 0$).

Proof. By Theorem A.2.6 and Theorem 4.1.3 of [10],

$$\mathbb{P}^y\{\sigma_{\breve{E}} = 0\} = 1 \text{ for q.e. } y \in \breve{E}.$$

Hence, for q.e. $y \in \breve{E}$ we have $\lim_{\epsilon \downarrow 0} \inf\{t > \epsilon : X_t \in \breve{E}\} = 0$, $\mathbb{P}^y$-a.s. Moreover, it follows from parts (i) and (ii) of Proposition 4.2 and the observation $\nu(\breve{E} \backslash \tilde{E}) = 0$ that for q.e. $y \in \breve{E}$ we have $\inf\{t > \epsilon : X_t \in \breve{E}\} = \inf\{t > \epsilon : X_t \in \tilde{E}\}$ for all $\epsilon > 0$, $\mathbb{P}^y$-a.s. Combining this with Proposition 6.1 gives

$$\mathbb{P}^y\{\sigma_{\tilde{E}} = 0\} = 1 \text{ for q.e. and } \nu\text{-a.e. } y \in \breve{E}.$$

Define $H_{\tilde{E}} f(x) := \mathbb{P}^x[f(X_{\sigma_{\tilde{E}}})]$ for f a bounded Borel function on E. It follows from part (i) of Proposition 4.2 and what we have just observed that

$$H_{\tilde{E}} f(x) = \frac{\int f(y) \kappa(x, y)\,\nu(dy)}{\int \kappa(x, y)\,\nu(dy)}, \text{ for } \mu\text{-a.e. x}$$

and

$$H_{\tilde{E}} f(x) = f(x), \text{ for } \nu\text{-a.e. x}.$$

The result now follows by applying Theorem 6.2.1 of [10]. □

Example 7.2. Suppose that we are in the setting of Example 4.7. For $y, z \in E^* = S^{\mathbb{N}_0}$, $y \neq z$, define $\delta(y, z) = \inf\{n : y_n \neq z_n\}$. Note that $\int \kappa(x, w)\,\nu(dw) = K(\zeta(x))\nu(\{w : x \leq w\}) = K(\zeta(x))\mu(\{x\})/R(\zeta(x))$ for $x \in E^o$ and so

$$\breve{\kappa}(y, z) = \sum_{n \leq \delta(y, z)} K(n) R(n). \tag{7.1}$$

We will now apply the results of Section 5 with $E, \mathbf{X}, \mu, \mathcal{E}$ replaced by $\breve{E} = E^* = S^{\mathbb{N}_0}, \mathbf{Y}, \nu, \breve{\mathcal{E}}$. Fix $N \in \mathbb{N}_0$ and let $\mathcal{R}'$ be the algebra of subsets of $S^{\mathbb{N}_0}$ of the form $B_0 \times \cdots \times B_N \times S \times S \times \ldots$. We can identify the quotient space $\bar{E}$ with S^{N+1} and the quotient map π with the map $(y_0, y_1, \ldots) \mapsto (y_0, \ldots y_N)$. Then we can identify $\bar{\mu}$ (which we emphasise is now the push–forward ν by π) with the measure that assigns mass $P(s_0)Q(s_0, s_1) \ldots Q(s_{N-1}, s_N)$ to $(s_0, \ldots s_N)$. Note that $\pi y \neq \pi z$ for $y, z \in S^{\mathbb{N}_0}$ is equivalent to $\delta(y, z) \leq N$, and it is immediate from (7.1) that Proposition 5.1 applies and $\pi \circ \mathbf{Y}$ is a $\bar{\mu}$-symmetric Markov chain on the finite state-space S^{N+1}. In terms of jump rates, $\pi \circ \mathbf{Y}$ jumps from $\bar{y}$ to $\bar{z} \neq \bar{y}$ at rate $(\sum_{n \leq \delta(\bar{y}, \bar{z})} K(n)R(n))\bar{\mu}(\{\bar{z}\})$, where $\delta(\bar{y}, \bar{z})$ is defined in the obvious way.

As a particular example of this construction, consider the case when $\#S = p^c$ for some prime p and integer $c \geq 1$. We can identify $S^{\mathbb{N}_0}$ (as a set) with the ring of integers $\mathbb{D}$ of a local field $\mathbb{K}$ as in Example 2.2. If we take $P(s) = Q(s, s') = p^{-c}$ for all $s, s' \in S$, then we can identify ν with the normalised Haar measure on $\mathbb{D}$. It is clear that $\mathbf{Y}$ is a Lévy process on $\mathbb{D}$ with "spherically symmetric" Lévy measure $\phi(|y|)\, \nu(dy)$, where $\phi(p^{-cn}) = \sum_{\ell=0}^{n} K(\ell)R(\ell)$. The condition $\sum_{n=0}^{\infty} K(n)R(n)p^{-cn} = \infty$ of Example 4.7 is equivalent to $\int_{\mathbb{D}} \phi(|y|)\, \nu(dy) = \infty$. Conversely, any Lévy process on $\mathbb{D}$ with Lévy measure of the form $\psi(|y|)\, \nu(dy)$ with ψ non-increasing and $\int_{\mathbb{D}} \psi(|y|)\, \nu(dy) = \infty$ can be produced by this construction (Lévy processes on $\mathbb{D}$ are completely characterised by their Lévy measures – there is no analogue of the drift or Gaussian components of the Euclidean case, see [7]). The latter condition is equivalent to the paths of the process almost surely not being step–functions, that is, to the times at which jumps occur being almost surely dense. When $\psi(|y|) = a|y|^{-(\alpha+1)}$ for some $a > 0$ and $0 < \alpha < \infty$, the resultant process is analogous to a symmetric stable process. Lévy processes on local fields and totally disconnected Abelian groups in general are considered in [7] and the special case of the p-adic numbers has been considered by a number of authors, including [1], [2], [9].

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E-mail address: aldous@stat.Berkeley.EDU

E-mail address: evans@stat.Berkeley.EDU

Department of Statistics #3860, University of California at Berkeley, 367 Evans Hall, Berkeley, CA 94720-3860, U.S.A