A lattice path model for the Bessel polynomials*

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Abstract

The $(n-1)$th Bessel polynomial is represented by an exponential generating function derived from the number of returns to 0 of a sequence with $2n$ increments of ± 1 which starts and ends at 0.

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It is well known [21, §3.71 (12)],[6, (7.2(40)] that the *McDonald function* or *Bessel function of imaginary argument*

$$K_\nu(x) := \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \int_0^\infty t^{\nu-1} e^{-t-(x/2)^2/t}\, dt \tag{1}$$

admits the evaluation

$$K_{n+1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\theta_n(x)x^{-n} \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

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where

$$\theta_n(x) := \sum_{m=0}^{n} \beta_{n,n-m} x^m \text{ with } \beta_{n,k} := \frac{(n+k)!}{2^k (n-k)! k!}. \tag{3}$$

The *Bessel polynomials*

$$\theta_n(x) \text{ and } y_n(x) := \sum_{k=0}^{n} \beta_{n,k} x^k = x^n \theta_n(x^{-1}) \tag{4}$$

have been extensively studied and applied: see the book of Grosswald [9] for a review. Dulucq and Favreau [4, 5] gave a combinatorial model for the Bessel polynomials based on the remark that

$$\beta_{n,k} = \binom{n+k}{n-k} \times (2k-1) \times (2k-3) \times \cdots \times 1$$

is the number of involutions of $n+k$ points with $n-k$ fixed points and k matched pairs of points forming 2-cycles. Their model is similar to a well known interpretation of the coefficients of the Hermite polynomials, which was extended to q-Hermite polynomials by Ismail, Stanton and Viennot [12]. Dulucq [3] treats a q-analog of the Bessel polynomials. See also Leroux and Strehl [13] for a model which interprets the coefficients of Jacobi polynomials, and Viennot [20] for other results in this vein.

The purpose of this note is to point out an alternative combinatorial model for the Bessel polynomials, based on an exponential generating function derived from lattice path enumerations. Call a sequence $b = (b_0, b_1, \ldots, b_{2n})$ a *lattice bridge of length* $2n$ if $b_0 = b_{2n} = 0$ and $b_i - b_{i-1} = \pm 1$ for every $1 \le i \le 2n$. Let B_n denote the set of all $\binom{2n}{n}$ lattice bridges of length $2n$. For $b \in B_n$ let $r(b)$ be the number of returns to 0 by b:

$$r(b) := \#\{i : 1 \le i \le 2n \text{ and } b_i = 0\}.$$

Then for each $n = 1, 2, \ldots$

$$\sum_{b \in B_n} \frac{x^{r(b)}}{r(b)!} = \frac{2^n}{n!} \, x \, \theta_{n-1}(x). \tag{5}$$

This formula can be read from [15, Corollary 9], which gives various probabilistic expressions of the formula in terms of random walks and Brownian motion. This approach connects formula (5) to the integral representation (1) of $K_{n-1/2}(x)$, and to formulae for generalized Stirling numbers due to Toscano [18, 19].

For $1 \le r \le n$ let $\#_{n,r}$ be the number of lattice bridges of length $2n$ with r returns to 0:

$$\#_{n,r} := \#\{b \in B_n : r(b) = r\}. \tag{6}$$

Then (5) amounts via (3) to the formula

$$\#_{n,r} = 2^n \frac{r!}{n!} \beta_{n-1,n-r} \tag{7}$$

which reduces to

$$\#_{n,r} = 2^r \binom{2n-r}{n} \frac{r}{2n-r}. \tag{8}$$

This can be read from Feller [7, III.7, Theorem 4]. Let $\#^+_{n,r}$ be the number of non-negative lattice bridges of length $2n$ with r returns to 0. Then (8) is equivalent to

$$\#^+_{n,r} = \binom{2n-r}{n} \frac{r}{2n-r}. \tag{9}$$

By the well known bijection of Harris [10] between between plane trees with n vertices and *lattice excursions of length* $2n$, that is non-negative lattice bridges b of length $2n$ with $r(b) = 1$, the number in (9) is the number of forests of r plane trees with n vertices [14, (6.1)]. The particular case $r = 1$ of (8) is the standard enumeration

$$\#_{n,1} = 2C_{n-1} \text{ where } C_n := \frac{1}{n+1}\binom{2n}{n} \tag{10}$$

is the nth Catalan number [7], [17, Cor. 6.2.3]. The corresponding generating function is well known to be

$$\sum_{n=1}^{\infty} \#_{n,1} w^n = 2\sum_{n=1}^{\infty} C_{n-1} w^n = 1 - (1-4w)^{1/2}. \tag{11}$$

It was already noted by Carlitz [1] that a number of results involving the Bessel polynomials acquire their simplest form when stated in terms of the polynomial $x\theta_{n-1}(x)$ which features in (5). In particular, Carlitz gave the exponential generating function

$$1 + \sum_{n=1}^{\infty} \frac{x\theta_{n-1}(x)}{2^n} \frac{u^n}{n!} = \exp[x(1-(1-u)^{1/2})]. \tag{12}$$

Formula (5) may be regarded as a combinatorial expression of the connection between the Bessel polynomials and the Catalan numbers implied by (12) and (11), exploiting (10) and the decomposition of a lattice path with r returns to 0 into its r excursions away from 0. To express this in terms of generating functions, observe from (5) that $\#_{n,r}$ is the coefficient of $\frac{x^r}{r!}$ in $\frac{2^n}{n!} x\theta_{n-1}(x)$. Symbolically, using the notation of [17],

$$\#_{n,r} = \left[\frac{x^r}{r!}\right] \frac{2^n}{n!} x\theta_{n-1}(x). \tag{13}$$

With similar notation, Carlitz's identity (12) can be restated as

$$\left[\frac{x^r}{r!}\right] 2^{-n} x\theta_{n-1}(x) = \left[\frac{u^n}{n!}\right] (1-(1-u)^{1/2})^r. \tag{14}$$

According to a classical expansion of Lambert [8, (5.70)]

$$[u^n](1-(1-u)^{1/2})^r = 2^{r-2n}\binom{2n-r}{n}\frac{r}{2n-r}. \tag{15}$$

Thus the form (14) of Carlitz's identity (12) can be read from (13), (8), and (15). Alternatively, the lattice path representation (5) of the Bessel polynomials could be deduced via (13) from (8), the form (14) of Carlitz's identity (12), and (15). See also Roman [16, p. 78] and Di Bucchianico [2, p. 54] for closely related discussions based on the consequence of (12) that the sequence of polynomials $f_n(x)$ is of binomial type.

Let

$$(z \mid \alpha)_n := \prod_{i=0}^{n-1}(z - i\alpha)$$

be the generalized factorial with decrement α, and let $S(n,k;\alpha,\beta)$ be the generalized Stirling numbers defined by

$$(z \mid \alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta)(z \mid \beta)_k.$$

In particular, the $S(n,k,1,0)$ and $S(n,k,0,1)$ are the classical Stirling numbers of the first and second kinds respectively. See Hsu and Shiue [11] for a recent review of the properties of these generalized Stirling numbers. According to [11, (14)] the polynomials

$$S_{n,\alpha,\beta}(x) := \sum_{k=0}^{n} S(n,k;\alpha,\beta)x^k$$

are determined for $\alpha \neq 0$ by the generating function

$$\sum_{n=0}^{\infty} S_{n,\alpha,\beta}(x)\frac{t^n}{n!} = \exp\left[\frac{x}{\beta}\left((1+\alpha t)^{\beta/\alpha} - 1\right)\right]. \tag{16}$$

Compare (12) and (16) for $\alpha = -2, \beta = -1$ to deduce that for all $n \geq 1$

$$S_{n,-2,-1}(x) = x\theta_{n-1}(x) \tag{17}$$

That is, from (3),

$$S(n,k,-2,-1) = \beta_{n-1,n-k} = \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}. \tag{18}$$

This expression for $S(n,k,-2,-1)$ is equivalent to a formula given without proof by Toscano [18, (122)],[19, (2.11)] along with several other explicit evaluations of generalized Stirling numbers. See also [15] for a probabilistic interpretation of the $S(n,k,-\alpha,-1)$ for arbitrary $\alpha > 1$ which yields asymptotic evaluations of these numbers for large n and k.

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