IMMANANTS AND FINITE POINT PROCESSES

PERSI DIACONIS AND STEVEN N. EVANS

Abstract. Given a Hermitian, non-negative definite kernel K and a character χ of the symmetric group on n letters, define the corresponding immanant function $K^\chi[x_1, \ldots, x_n] := \sum_\sigma \chi(\sigma) \prod_{i=1}^n K(x_i, x_{\sigma(i)})$, where the sum is over all permutations σ of $\{1, \ldots, n\}$. When χ is the sign character (resp. the trivial character), then K^χ is a determinant (resp. permanent). The function K^χ is symmetric and non-negative, and, under suitable conditions, is also non-trivial and integrable with respect to the product measure $\mu^{\otimes n}$ for a given measure μ. In this case, K^χ can be normalised to be a symmetric probability density. The determinantal and permanental cases or this construction correspond to the fermion and boson point processes which have been studied extensively in the literature.

The case where K gives rise to an orthogonal projection of $L^2(\mu)$ onto a finite–dimensional subspace is studied here in detail. The determinantal instance of this special case has a substantial literature because of its role in several problems in mathematical physics, particularly as the distribution of eigenvalues for various models of random matrices. The representation theory of the symmetric group is used to compute the normalisation constant and identify the k^{th}–order marginal densities for $1 \le k \le n$ as linear combinations of analogously defined immanantal densities. Connections with inequalities for immanants, particularly the permanental dominance conjecture of Lieb, are considered, and asymptotics when the dimension of the subspace goes to infinity are presented.

1. Introduction

Gian–Carlo Rota loved symmetric functions and probability. This paper brings these two subjects together.

Consider a σ–finite measure space $(\Sigma, \mathcal{A}, \mu)$ with the measure μ diffuse. Suppose that $K \in L^2(\mu \otimes \mu)$ is a non–negative definite, Hermitian kernel on Σ with finite trace. That is,

$$K(x, y) = \bar{K}(y, x), \tag{1.1}$$

$$\sum_{i,j} \bar{z}_i K(x_i, x_j) z_j \ge 0, \quad z_1, \ldots, z_n \in \mathbb{C}, \ x_1, \ldots x_n \in \Sigma, \tag{1.2}$$

and

$$\int K(x, x)\mu(dx) < \infty. \tag{1.3}$$

Date: March 20, 2000.

1991 *Mathematics Subject Classification.* Primary: 15A15, 60G55; Secondary: 20C30, 60G09.

Key words and phrases. symmetric group, character, immanant, determinant, permanent, point process, fermion process, boson process.

Research supported in part by NSF grant DMS-9504379.

Research supported in part by NSF grant DMS-9703845.

Given a partition β of n, let χ^β be the character of the corresponding irreducible representation of the symmetric group on n letters, $\mathfrak{S}_n$ (see, for example, Ch. 4 of [FH91] or Ch. VI of [Sim96]). Given $x_1, \ldots, x_n \in \Sigma$, write $K^\beta[x_1, \ldots, x_n]$ for the *immanant* corresponding to χ^β of the matrix with ij^{th} entry $K(x_i, x_j)$ (see Ch. VI of [Lit58] and [Jam87, Jam92]). That is,

$$K^\beta[x_1, \ldots, x_n] := \sum_{\sigma \in \mathfrak{S}_n} \chi^\beta(\sigma) \prod_{i=1}^n K(x_i, x_{\sigma(i)}).$$

Note that if $\tau \in \mathfrak{S}_n$, then

$$\begin{aligned} K^\beta[x_{\tau(1)}, \ldots, x_{\tau(n)}] &= \sum_{\sigma \in \mathfrak{S}_n} \chi^\beta(\sigma) \prod_{i=1}^n K(x_{\tau(i)}, x_{\tau(\sigma(i))}) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \chi^\beta(\sigma) \prod_{i=1}^n K(x_i, x_{\tau\sigma\tau^{-1}(i)}) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \chi^\beta(\tau^{-1}\sigma\tau) \prod_{i=1}^n K(x_i, x_{\sigma(i)}) \\ &= K^\beta[x_1, \ldots, x_n], \end{aligned}$$

because $\chi^\beta(\tau^{-1}\sigma\tau) = \chi^\beta(\sigma)$ (that is, χ^β is a *class function*). In other words, K^β is a symmetric function.

It follows from the Cauchy–Schwarz inequality that

$$\left|\prod_{i=1}^n K(x_i, x_{\sigma(i)})\right| \le \prod_{i=1}^n K(x_i, x_i)^{\frac{1}{2}} K(x_{\sigma(i)}, x_{\sigma(i)})^{\frac{1}{2}} = \prod_{i=1}^n K(x_i, x_i) \tag{1.4}$$

for any permutation σ, and so (1.3) implies that K^β is integrable with respect to $\mu^{\otimes n}$. By a result of Schur [Sch18] (see also [Jam87]),

$$K^\beta[x_1, \ldots, x_n] \ge \chi^\beta(1) \det(K(x_i, x_j)) \ge 0. \tag{1.5}$$

Therefore, when $K^\beta > 0$ on a set of positive $\mu^{\otimes n}$–measure the function K^β can be renormalised to be the n^{th} Janossy measure density (with respect to $\mu^{\otimes n}$) of a finite simple point process on Σ with exactly n points. Informally, for some constant $c_{K,\beta}$ the quantity $c_{K,\beta} K^\beta[x_1, \ldots, x_n]\, \mu(dx_1) \ldots \mu(dx_n)$ is the probability that a realisation of the point process will result in one point located in each of the the infinitesimal subsets dx_i and no points elsewhere. In particular,

$$\int \cdots \int c_{K,\beta} K^\beta[x_1, \ldots, x_n]\, \mu(dx_1) \ldots \mu(dx_n) = n!.$$

When $\beta = (1^n)$ (that is, the partition consisting of n parts which are all 1) we have that $\chi^\beta(\sigma) = \operatorname{sgn}(\sigma)$, the sign of the permutation σ, and $K^\beta[x_1, \ldots, x_n] = \det(K(x_i, x_j))$. When $\beta = (n)$ (that is, the partition consisting of a single part n) we have that $\chi^\beta \equiv 1$ and $K^\beta[x_1, \ldots, x_n] = \operatorname{per}(K(x_i, x_j))$, the *permanent* of the matrix $(K(x_i, x_j))$. The corresponding point processes are discussed in [Mac75], where they called, respectively, *fermion* and *boson* processes on account of their origins in quantum mechanics (see also [DVJ88]). The physical terminology *fermion* is suggestive of the Pauli exclusion principle, and it is indeed the case that such processes exhibit "antibunching" effects which are absent in the boson case. A recent survey (with an extensive bibliography) of the fermion case and its role in quantum mechanics, statistical mechanics, random matrix theory, representation theory, and ergodic theory may be found in [Sos00]. The point processes for general characters, which don't appear to have been mentioned previously in the literature, can be thought of as "interpolating" between the fermion and boson cases.

The point processes of eigenvalues for various models of random matrices turn out to be fermion processes (see [Meh91]). In these examples, the kernel K corresponds to an orthogonal projection $\mathcal{P}_S$ onto a finite–dimensional subspace S of $L^2(\mu)$. That is, $\mathcal{P}_S f(x) = \int K(x,y) f(y)\,\mu(dy)$. As a projection, the function K has the extra properties:

$$\int K(x,y)K(y,z)\,\mu(dy) = K(x,z) \tag{1.6}$$

and

$$\int K(x,x)\mu(dx) = \dim S =: D_S. \tag{1.7}$$

Moreover, if $\{\varphi_i : 1 \le i \le \dim S\}$ is an orthonormal basis for S, then

$$K(x,y) = \sum_i \varphi_i(x)\bar{\varphi}_i(y). \tag{1.8}$$

For example, consider a uniformly chosen random $N \times N$ unitary matrix (that is, a random matrix distributed according to Haar measure on the unitary group). The point process on the unit circle formed by the N eigenvalues of such a matrix has N^{th} Janossy measure density against Lebesgue measure given by $\det(S_N(\theta_j - \theta_k))$, where

$$\begin{aligned} S_N(\theta) &:= \frac{1}{2\pi} \frac{\sin\left(\frac{N\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\ &= e^{-i\frac{N-1}{2}\theta}\left(1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{i(N-1)\theta}\right). \end{aligned}$$

Here, of course, we are identifying the unit circle with the interval $[0, 2\pi[$ and the Lebesgue measure has total mass 2π.

From know on we will consider special case of projection kernels and write K_S for K to stress the dependence on the subspace S. To simplify notation we will write $c_{S,\beta}$ for the normalisation constant $c_{K_S,\beta}$.

It is apparent from the random matrix examples in [Meh91] (see, particularly, Theorem 5.2.1) that fermion processes corresponding to projection kernels share the useful property that it is possible to evaluate the necessary integrals to compute the normalisation constant $c_{S,(1^n)}$ explicitly and to find the corresponding k^{th}*–order factorial moment measure densities*

$$m_{[k]}^{S,(1^n)}(x_1, \ldots, x_n) := \int \cdots \int \frac{1}{(n-k)!} c_{S,\beta} K_S^{(1^n)}[x_1, \ldots, x_n]\,\mu(dx_{k+1}) \ldots \mu(dx_n)$$

for $1 \le k < n$. The quantity $m_{[k]}^{S,(1^n)}(x_1, \ldots, x_n)\,\mu(dx_1) \ldots \mu(dx_k)$ is the probability that a realisation of the fermion process will result in one point located in each of the the infinitesimal subsets dx_i (with no constraints on the remaining $n-k$ points). Alternatively, $\binom{n}{k}^{-1} m_{[k]}^{S,(1^n)}$ is k^{th} Janossy measure of the point process obtained by picking k points at random from the original n points laid down by the fermion process. In the physics terminology used in the random matrix literature, $m_{[k]}^{S,(1^n)}$ is the *k–point correlation function* for the fermion process.

It is our aim here to use the representation theory of $\mathfrak{S}_n$ to show that analogues of explicit integration formulae for determinants of projection kernels hold for general immanants. Also, we will relate such integration formulae to the extensive literature on inequalities for immanants which has grown out of the *permanental dominance conjecture* of Lieb [Lie66] (see also [Mer87, Jam87, Jam92, Pat94, Pat98, Pat99] for surveys and extensive bibliographies). Finally, we consider the asymptotics of the point process when the dimension D_S goes to infinity.

We end this section with some further comments on the immanants literature and its relation to our work.

Computationally, determinants are known to be "easy" to evaluate, whereas permanents are "hard" (see the seminal paper [Val79] and the recent review [Cla96]). However, there are good randomised algorithms for approximating permanents (see, for example, [Bar97, Bar99]). Upper bounds on the computational complexity of general immanants are discussed in [Har85, Bar90]. An efficient algorithm for evaluating the immanant when the character of $\mathfrak{S}_n$ corresponds to the partition $(2, 1^{n-2})$ is presented in [GM84], where the use of immanants in constructing graph invariants is also discussed.

Lastly, we note that if the kernel K is no longer Hermitian but is such that the matrix $(K(x_i, x_j))$ is totally positive for all $x_1, \ldots, x_n$ (that is, all minors are non–negative), then an analogue of Schur's inequality due to Stembridge [Ste91] holds and so it is again possible under suitable integrability conditions to construct for an arbitrary partition β a finite point process with n^{th} Janossy measure density K^β.

2. Integration formulae

As usual, we associate partitions of n with Young frames using the convention of, say, [FH91] or [Sim96]. That is, the Young frame associated with a partition $\beta = (\beta_1, \ldots, \beta_k)$ with $\beta_1 \geq \cdots \geq \beta_k \geq 1$ consists of k "left–justified" rows of boxes, where the top row has β_1 boxes, the second row has β_2 boxes, and so on.

Thinking of two partitions α of $n-1$ and β of n as Young frames, say that $\alpha \lhd \beta$ if α is obtained from β by the removal of a boundary box (that is, a box at the right–hand end of a row of β). Note that the box to be removed is also at the bottom of a column of β. In this case, write $M^{\nearrow}(\alpha, \beta)$ for the length of the *hook* in β that contains the removed box and the *rightmost* box in the *top* row of β. Similarly, write $M^{\swarrow}(\alpha, \beta)$ for the length of the hook in β that contains the removed box and the *leftmost* box in the *bottom* row of β. That is, if we write $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_\ell > 0$ and $\alpha_h = \beta_h$ for all indices h except for one index k for which $\alpha_k = \beta_k - 1$, then $M^{\nearrow}(\alpha, \beta) = k + \beta_1 - \beta_k$ and $M^{\swarrow}(\alpha, \beta) = (\ell - k) + \beta_k$.

Note that if α' and β' denote the conjugates of α and β, then $\alpha \lhd \beta$ if and only if $\alpha' \lhd \beta'$, in which case $M^{\nearrow}(\alpha, \beta) = M^{\swarrow}(\alpha', \beta')$ and $M^{\swarrow}(\alpha, \beta) = M^{\nearrow}(\alpha', \beta')$. Note also that $M^{\nearrow}(\alpha, \beta)$ (resp. $M^{\swarrow}(\alpha, \beta)$) is the length of the *skew hook* in β that contains the removed box and the rightmost box in the top row (resp. leftmost box in the bottom row) of β (recall that a skew hook is a connected chain of boundary boxes).

Theorem 2.1. *Let $\beta = (\beta_1, \beta_2, \dots, \beta_k)$ be a partition of $n \geq 2$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_k > 0$.*

(a) In the notation above,

$$\int_\Sigma K_S^\beta[x_1, \dots, x_n]\, \mu(dx_n) = \sum_{\alpha \triangleleft \beta} \left(D_S - M^{\nearrow}(\alpha, \beta) + M^{\swarrow}(\alpha, \beta)\right) K_S^\alpha[x_1, \dots, x_{n-1}].$$

(b) Write $\ell_1 = \beta_1 + k - 1$, $\ell_2 = \beta_2 + k - 2$, $\dots$, $\ell_k = \beta_k$. Then

$$\frac{1}{n!} \int_{\Sigma^n} K_S^\beta[x_1, \dots, x_n]\, \mu^{\otimes n}(dx)$$

is the coefficient of $u^n y_1^{\ell_1} y_2^{\ell_2} \dots y_k^{\ell_k}$ in

$$\prod_{1 \leq a < b \leq k} (y_a - y_b) \cdot \prod_{c=1}^k (1 - u y_c)^{-D_S},$$

which is

$$\begin{aligned} &(-1)^n \sum_\sigma (\operatorname{sgn} \sigma) \prod_{c=1}^k \binom{-D_S}{\ell_c - \sigma(k+1-c) + 1} \\ &= \sum_\sigma (\operatorname{sgn} \sigma) \prod_{c=1}^k \binom{D_S + \ell_c - \sigma(k+1-c)}{\ell_c - \sigma(k+1-c) + 1}, \end{aligned}$$

where the sum is over all permutations $\sigma \in \mathfrak{S}_k$ such that $\sigma(k+1-c) \leq \ell_c + 1$ for $1 \leq c \leq k$.

Proof. (a) Identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ that fixes n. For $1 \leq k \leq n-1$ write (kn) for the element of $\mathfrak{S}_n$ which transposes k and n and leaves all other elements of $\{1, \dots, n\}$ fixed. If $\tau \in \mathfrak{S}_{n-1}$, then the product $\tau(kn)$ is a permutation which has the effect $k \to n \to \tau(k)$ and $(\tau(kn))(i) = \tau(i)$ for $i \notin \{k, n\}$. We have

(2.1)
$$\begin{aligned} &\int K_S^\beta[x_1, \dots, x_n]\, \mu(dx_n) \\ &= \sum_{\tau \in \mathfrak{S}_{n-1}} \chi^\beta(\tau) \prod_{i=1}^{n-1} K_S(x_i, x_{\tau(i)}) \int K_S(x_n, x_n)\, \mu(dx_n) \\ &\quad + \sum_{\tau \in \mathfrak{S}_{n-1}} \sum_{k=1}^{n-1} \chi^\beta(\tau(kn)) \prod_{i=1, i \neq k}^{n-1} K_S(x_i, x_{\tau(i)}) \int K_S(x_k, x_n) K_S(x_n, x_{\tau(k)})\, \mu(dx_n). \end{aligned}$$

Now χ^β restricted to $\mathfrak{S}_{n-1}$ is just the character of the restricted representation and so, by the usual *branching rule* (see, for example Exercise 4.43 of [FH91]),

$$\chi^\beta(\tau) = \sum_{\alpha \triangleleft \beta} \chi^\alpha(\tau).$$

By (1.7) the first sum in the right side of (2.1) is thus

$$\sum_{\alpha \triangleleft \beta} D_S K_S^\alpha[x_1, \dots, x_{n-1}]. \tag{2.2}$$

Turning to the second sum on the right side of (2.1), note from (1.6) that

$$\prod_{i=1, i\neq k}^{n-1} K_S(x_i, x_{\tau(i)}) \int K_S(x_k, x_n) K_S(x_n, x_{\tau(k)})\, \mu(dx_n) = \prod_{i=1}^{n-1} K_S(x_i, x_{\tau(i)}).$$

Note also that because χ^β is a class function on $\mathfrak{S}_n$, the function $\tau \mapsto \sum_{k=1}^{n-1} \chi^\beta(\tau(kn))$ is a class function on $\mathfrak{S}_{n-1}$. Therefore, there exist constants $C_{\alpha,\beta}$ such that $\sum_{k=1}^{n-1} \chi^\beta(\tau(kn)) = \sum_\alpha C_{\alpha,\beta}\chi^\alpha(\tau)$, where the sum on the right is over all partitions α of $n-1$. Thus

$$\begin{aligned}\sum_{\tau\in\mathfrak{S}_{n-1}} \sum_{k=1}^{n-1} \chi^\beta(\tau(kn)) \prod_{i=1, i\neq k}^{n-1} K_S(x_i, x_{\tau(i)}) \int K_S(x_k, x_n) K_S(x_n, x_{\tau(k)})\, \mu(dx_n) \\ = \sum_\alpha C_{\alpha,\beta} K_S^\alpha[x_1, \ldots, x_{n-1}].\end{aligned}$$

By orthogonality of characters

$$C_{\alpha,\beta} = \frac{1}{(n-1)!} \sum_{\tau\in\mathfrak{S}_{n-1}} \sum_{k=1}^{n-1} \chi^\beta(\tau(kn))\chi^\alpha(\tau^{-1}).$$

Suppose first of all that $\alpha \triangleleft \beta$. Fix for the moment $\tau \in \mathfrak{S}_{n-1}$ and $1 \le k \le n-1$. The cycle decomposition of $\tau(kn)$ consists of a cycle π of length m, say, that contains the sequence $\cdots \to k \to n \to \tau(k) \to \ldots$ and a collection of cycles that we denote by υ. The cycle decomposition of τ consists of the collection υ and a cycle ρ of length $m-1$ that agrees with π except that the sequence $\cdots \to k \to n \to \tau(k) \to \ldots$ is replaced by the sequence $\cdots \to k \to \tau(k) \to \ldots$.

By the Murnaghan–Nakayama rule (see, for example, Problem 4.45 in ([FH91])), we have

$$\chi^\beta(\tau(kn)) = \sum_\delta (-1)^{r(\delta,\beta)}\chi^\delta(\upsilon),$$

where the sum is over all Young frames δ of size $n-m$ obtained by removing a skew hook of length m from β and $r(\delta,\beta)$ is the number of vertical steps in the skew hook (that is, one less than the number of rows in the skew hook). Here, of course, we are viewing the collection of cycles υ as the cycle decomposition of a permutation on the $n-m$ elements of $\{1,\ldots,n\}$ not contained in the cycle π (equivalently, as the cycle decomposition of a permutation of the $n-m$ elements of $\{1,\ldots,n-1\}$ not contained in the cycle ρ), and hence as an element of $\mathfrak{S}_{n-m}$. Similarly,

$$\chi^\beta(\tau^{-1}) = \sum_\gamma (-1)^{r(\gamma,\alpha)}\chi^\gamma(\upsilon^{-1}),$$

where the sum is over all Young frames γ of size $(n-1)-(m-1) = n-m$ obtained by removing a skew hook of length $m-1$ from α and $r(\gamma,\alpha)$ has the obvious meaning.

Fix for the moment γ and δ such that γ is obtained by removing a skew hook of length $m-1$ from α and δ is obtained by removing a skew hook of length m from β. By the orthogonality of characters, if ρ is, as above, a fixed $(m-1)$–cycle drawn from $\{1, \ldots, n-1\}$ which contains k, then

$$\sum_{\upsilon} \chi^\delta(\upsilon)\chi^\gamma(\upsilon^{-1}) = \begin{cases} (n-m)!, & \text{if } \gamma = \delta, \\ 0, & \text{otherwise,} \end{cases}$$

where the sum is over all permutations υ of the $n-m$ letters not contained in the $(m-1)$–cycle ρ. Now $\gamma = \delta$ if and only if the skew hook of length m removed from β has the box that needs to be removed from β to obtain α as either its "northeast-most" box, in which case $(-1)^{r(\gamma,\alpha)} = (-1)^{r(\delta,\beta)}$, or its "southwest–most" box, in which case $(-1)^{r(\gamma,\alpha)} = -(-1)^{r(\delta,\beta)}$.

Therefore

$$\begin{aligned}&\sum_{\upsilon}\sum_{\delta}\sum_{\gamma}(-1)^{r(\delta,\beta)}(-1)^{r(\gamma,\alpha)}\chi^\delta(\upsilon)\chi^\gamma(\upsilon^{-1}) \\ &\quad = (n-m)!(I_m^{\swarrow}(\alpha,\beta) - I_m^{\nearrow}(\alpha,\beta)),\end{aligned}$$

where $I_m^{\swarrow}(\alpha,\beta) = 1$ if the box that needs to be removed from β to obtain α is the "northeast-most" box in a skew hook of length m and $I_m^{\swarrow}(\alpha,\beta) = 0$ otherwise, and $I_m^{\nearrow}(\alpha,\beta) = 1$ if the box that needs to be removed from β to obtain α is the "southwest-most" box in a skew hook of length m and $I_m^{\nearrow}(\alpha,\beta) = 0$ otherwise.

For each $1 \le k \le n-1$ the number of $(m-1)$–cycles drawn from $\{1, \ldots, n-1\}$ which contain k is $(n-2)!/(n-m)!$. Therefore,

$$\begin{aligned} C_{\alpha,\beta} &= \frac{1}{(n-1)!}(n-1)\sum_{m=2}^{n-1}\frac{(n-2)!}{(n-m)!}(n-m)!(I_m^{\swarrow}(\alpha,\beta) - I_m^{\nearrow}(\alpha,\beta)) \\ &= M^{\swarrow}(\alpha,\beta) - M^{\nearrow}(\alpha,\beta)\end{aligned}$$

A similar argument shows that $C_{\alpha,\beta} = 0$ if $\alpha \triangleleft \beta$ does not hold, and this completes the proof of part (a).

(b) Given $\sigma \in \mathfrak{S}_n$, write $\#(\sigma)$ for the number of cycles in σ. It follows from (1.6) and (1.7) that

$$\int_{\Sigma^n}\prod_{i=1}^n K_S(x_i, x_{\sigma(i)})\,\mu^{\otimes n}(dx) = D_S^{\#(\sigma)},$$

and so

$$\frac{1}{n!}\int_{\Sigma^n} K_S^\beta[x_1, \ldots, x_n]\,\mu^{\otimes n}(dx) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\chi^\beta(\sigma)D_S^{\#(\sigma)}. \tag{2.3}$$

Set

$$\Delta(y) := \prod_{1\le a<b\le k}(y_a - y_b)$$

and

$$P_j(y) := y_1^j + y_2^j + \cdots + y_k^j, \quad 1 \le j \le k.$$

By the Frobenius character formula (see, for example, §4.1 of [FH91]), if σ has i_1 1–cycles, i_2 2–cycles, …, i_n n–cycles, then $\chi^\beta(\sigma)$ is the coefficient of $y_1^{\ell_1}y_2^{\ell_2}\ldots y_k^{\ell_k}$ in

$$\Delta(y)\cdot\prod_{j=1}^n P_j(y)^{i_j},$$

Note that the number of elements of $\mathfrak{S}_n$ with cycle structure $(i_1, i_2, \ldots, i_n)$ is

$$\frac{n!}{1^{i_1} i_1! 2^{i_2} i_2! \ldots n^{i_n} i_n!}$$

(see equation (4.30) of [FH91]). Moreover,

$$\sum_{i_1, \ldots i_n} \prod_{j=1}^{n} t_j^{i_j} \frac{1}{i_1! i_2! \ldots i_n!}$$

(where the sum is over all $i_1, i_2, \ldots, i_n$ such that $\sum_j j i_j = n$) is the coefficient of u^n in

$$\exp\left(\sum_{j=1}^{n} t_j u^j\right).$$

Therefore, the right–hand side of (2.3) is the coefficient of $u^n y_1^{\ell_1} y_2^{\ell_2} \ldots y_k^{\ell_k}$ in

$$\begin{aligned} \Delta(y) \cdot \exp\left(\sum_{j=1}^{n} \frac{D_S P_j(y)}{j} u^j\right) &= \Delta(y) \cdot \exp\left(\sum_{j=1}^{n} \frac{D_S}{j} u^j \left(\sum_{c=1}^{k} y_c^j\right)\right) \\ &= \Delta(y) \cdot \exp\left(\sum_{c=1}^{k} D_S \sum_{j=1}^{n} \frac{(u y_c)^j}{j}\right), \end{aligned}$$

which is in turn the coefficient of $u^n y_1^{\ell_1} y_2^{\ell_2} \ldots y_k^{\ell_k}$ in

$$\begin{aligned} \Delta(y) \cdot \prod_{c=1}^{k} \exp\left(D_S \sum_{j=1}^{\infty} \frac{(u y_c)^j}{j}\right) &= \Delta(y) \cdot \prod_{c=1}^{k} \exp\left(-D_S \log(1 - u y_c)\right) \\ &= \Delta(y) \cdot \prod_{c=1}^{k} (1 - u y_c)^{-D_S}. \end{aligned}$$

The proof of part (b) is completed by noting that $\Delta(y)$ is the Vandermonde determinant

$$\det \begin{pmatrix} 1 & y_k & \cdots & y_k^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_1 & \cdots & y_1^{k-1} \end{pmatrix}$$

and that $\sum_{c=1}^{k} (\ell_c - \sigma(k+1-c) + 1) = n$. □

Combining part (b) of Theorem 2.1 with repeated applications of part (a) gives the following result.

Corollary 2.2. *Suppose that $\beta = (\beta_1, \ldots, \beta_k)$ is a partition of n as in Theorem 2.1. Then*

$$\begin{aligned} &\frac{1}{n!} \sum_{(1) = \alpha_1 \triangleleft \cdots \triangleleft \alpha_n = \beta} D_S \prod_{i=1}^{n-1} \left(D_S - M^{\nearrow}(\alpha_i, \alpha_{i+1}) + M^{\swarrow}(\alpha_i, \alpha_{i+1})\right) \\ &= \sum_{\sigma} (\operatorname{sgn} \sigma) \prod_{c=1}^{k} \binom{D_S + \ell_c - \sigma(k+1-c)}{\ell_c - \sigma(k+1-c) + 1}, \end{aligned}$$

where $\ell_1, \dots, \ell_k$ are as in Theorem 2.1 and the sum on the right–hand side is over all permutations $\sigma \in \mathfrak{S}_k$ such that $\sigma(k+1-c) \leq \ell_c + 1$ for $1 \leq c \leq k$.

Example 2.3. Suppose that β is a hook partition of the form $(m, 1^{n-m})$ for $1 \leq m \leq n$ (that is, the first row of β thought of a Young frame has m boxes and the remaining $n-m$ rows each have one box). Then

$$\begin{aligned}\int_\Sigma K_S^\beta[x_1, \dots, x_n]\,\mu(dx_n) &= (D_S - 1 + n)\,K_S^{((m-1),1^{n-m})}[x_1, \dots, x_{n-1}] \\ &\quad + (D_S - n + 1)\,K_S^{(m,1^{n-m-1})}[x_1, \dots, x_{n-1}]\end{aligned}$$

if $1 < m < n$, with the obvious modifications if $m = 1$ or $m = n$. Continuing in this way gives

$$\begin{aligned}&\int \cdots \int K_S^\beta[x_1, \dots, x_n]\,\mu(dx_{h+1}) \dots \mu(dx_n) \\ &= \sum (D_S + \varepsilon_{n-1}^{a,b}(n-1))(D_S + \varepsilon_{n-2}^{a,b}(n-2)) \dots (D_S + \varepsilon_h^{a,b} h) K_S^{(a,1^b)}[x_1, \dots, x_h],\end{aligned}$$

where the sum is over all $1 \leq a \leq m$ and $0 \leq b \leq n-m$ with $a + b = h$ and all $\varepsilon_{n-1}^{a,b}, \varepsilon_{n-2}^{a,b}, \dots, \varepsilon_h^{a,b} \in \{\pm 1\}$ such that $(m-a)$ of these terms are $+1$ and the remaining $(n-m)-b$ are -1. Equivalently, the term on the right–hand side is the coefficient of v^{m-a} in

$$\prod_{g=h+1}^{n} (D_S - (g-1) + v(D_S + (g-1)))\,.$$

In particular, considering the case $h = 1$ and then doing one more integration using (1.7) gives that

$$\begin{aligned}&\frac{1}{n!} \sum (D_S + \varepsilon_{n-1}^{1,0}(n-1))(D_S + \varepsilon_{n-2}^{1,0}(n-2)) \dots (D_S + \varepsilon_1^{1,0} 1) D_S \\ &\quad = \sum_\sigma (\operatorname{sgn}\sigma) \binom{D_S + n - \sigma(n-m+1)}{n - \sigma(n-m+1) + 1} \binom{D_S + n - m - \sigma(n-m)}{n - m - \sigma(n-m) + 1} \\ &\qquad \times \binom{D_S + n - m - 1 - \sigma(n-m-1)}{n - m - 1 - \sigma(n-m-1) + 1} \dots \binom{D_S + 1 - \sigma(1)}{1 - \sigma(1) + 1},\end{aligned}$$

where the sum on the left–hand side is over all $\varepsilon_{n-1}^{1,0}, \varepsilon_{n-2}^{1,0}, \dots, \varepsilon_h^{1,0} \in \{\pm 1\}$ such that $(m-1)$ of these terms are $+1$ and the remaining $(n-m)$ are -1, and the sum on the right–hand side is over all permutations $\sigma \in \mathfrak{S}_{n-m+1}$ such that $\sigma(c) \leq c+1$ for $1 \leq c \leq n-m+1$. For example, when $m = 1$ (so that the immanant is a determinant) this equality becomes

$$\begin{aligned}\binom{D_S}{n} &= \sum_\sigma (\operatorname{sgn}\sigma) \binom{D_S + n - \sigma(n)}{n + 1 - \sigma(n)} \binom{D_S + n - 1 - \sigma(n-1)}{n - \sigma(n-1)} \\ &\quad \times \binom{D_S + n - 2 - \sigma(n-2)}{n - 1 - \sigma(n-2)} \dots \binom{D_S + 1 - \sigma(1)}{2 - \sigma(1)},\end{aligned}$$

where the sum on the right–hand side is over all permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(c) \leq c + 1$ for $1 \leq c \leq n$.

3. Connections with immanant inequalities

The permanental dominance conjecture of Lieb [Lie66] asserts that $K^\beta \le \chi^\beta(e)K^{(n)}$ for any K satisfying (1.1) and (1.2), where e is the identity permutation. A consequence of this conjecture would therefore be that $K_S^\beta \le \chi^\beta(e)K_S^{(n)}$ and hence, in particular,

$$\begin{aligned}&\int\cdots\int K_S^\beta[x_1,\ldots,x_n]\,\mu(dx_1)\ldots\mu(dx_n)\\ &\quad\le \chi^\beta(e)\int\cdots\int K_S^{(n)}[x_1,\ldots,x_n]\,\mu(dx_1)\ldots\mu(dx_n).\end{aligned}\tag{3.1}$$

By (2.3) the left–hand side of the (3.1) is

$$\sum_{\sigma\in\mathfrak{S}_n}\chi^\beta(\sigma)D_S^{\#(\sigma)},$$

whereas the right–hand side is

$$\chi^\beta(e)\sum_{\sigma\in\mathfrak{S}_n}D_S^{\#(\sigma)},$$

and (3.1) does indeed hold because $|\chi^\beta(\sigma)|\le\chi^\beta(e)$ for all $\sigma\in\mathfrak{S}_n$.

A remarkable inequality of Pate [Pat92] gives a comparison of two immanants in which one partition is obtained from another by moving a corner box of the corresponding Young frame to the bottom of the frame. More precisely, suppose that $\beta=(\beta_1,\beta_2,\ldots,\beta_k)$ is a partition of n such that $\beta_1>1$. Suppose that $1\le h\le k$ is such that $\beta_h>\max(\beta_{h+1},1)$. Let β' denote the partition $(\beta_1,\ldots,\beta_{h-1},\beta_h-1,\beta_{h+1},\ldots,\beta_k,1)$. Then

$$K^\beta/\chi^\beta(e)\ge K^{\beta'}/\chi^{\beta'}(e).\tag{3.2}$$

The special case of this result for hook partitions was proved in [Hey88] and implies the validity of the permanental dominance conjecture for such partitions.

Applying (3.2) to $K=K_S$ and integrating, we find from (2.3) that

$$\sum_{\sigma\in\mathfrak{S}_n}\frac{\chi^\beta(\sigma)}{\chi^\beta(e)}d^{\#(\sigma)}\ge\sum_{\sigma\in\mathfrak{S}_n}\frac{\chi^{\beta'}(\sigma)}{\chi^{\beta'}(e)}d^{\#(\sigma)}$$

for all positive integers d.

Let $\ell_1>\ell_2>\ldots>\ell_k$ correspond to β as in Theorem 2.1, and define $\ell_1'>\ell_2'>\ldots>\ell_k'>\ell_{k+1}'=1$ analogously for β' so that $\ell_h'=\ell_h$ and $\ell_i'=\ell_i+1$ for $1\le i\le k$, $i\ne h$. Recall that

$$\chi^\beta(e)=\frac{n!}{\ell_1!\ldots\ell_k!}\prod_{i<j}(\ell_i-\ell_j),$$

with an analogous formula for $\chi^{\beta'}(e)$ (see (4.11) of [FH91]). It follows from Pate's inequality and Theorem 2.1 that for all positive integers d,

$$\begin{aligned}&\frac{\ell_1!\ldots\ell_k!}{\prod_{i<j}(\ell_i-\ell_j)}\sum_\sigma(\operatorname{sgn}\sigma)\prod_{c=1}^k\binom{d+\ell_c-\sigma(k+1-c)}{\ell_c-\sigma(k+1-c)+1}\\ &\ge\frac{\ell_1'!\ldots\ell_k'!}{\prod_{i<j}(\ell_i'-\ell_j')}\sum_{\sigma'}(\operatorname{sgn}\sigma')\prod_{c'=1}^{k+1}\binom{d+\ell_{c'}'-\sigma'(k+2-c')}{\ell_{c'}'-\sigma'(k+2-c')+1},\end{aligned}$$

where the sum on the left is over all permutations $\sigma \in \mathfrak{S}_k$ such that $\sigma(k+1-c) \leq \ell_c + 1$ for $1 \leq c \leq k$, and the sum on the right is over all permutations $\sigma' \in \mathfrak{S}_{k+1}$ such that $\sigma'(k+2-c') \leq \ell'_{c'} + 1$ for $1 \leq c' \leq k+1$.

4. Point process asymptotics

Write $\Pi_{S,\beta}$ for the point process with n^{th} Janossy measure K_S^β, where β is a partition of n.

Proposition 4.1. *Suppose that μ is finite and $\{S_m\}_{m\in\mathbb{N}}$ is a sequence of finite–dimensional subspaces of $L^2(\mu)$ with the property that*

$$\lim_{m\to\infty} \int \left| D_{S_m}^{-1} K_{S_m}(x,x) - \kappa(x) \right| \mu(dx) = 0$$

for some probability density κ. Then for any partition β of n the point processes $\Pi_{S_m,\beta}$ converge in total variation as $m \to \infty$ to the point process obtained by laying down n independent draws from the distribution with density κ.

Proof. Assume without loss of generality that μ is a probability measure. For ease of notation, write K_m for K_{S_m}, D_m for D_{S_m}, and $c_{m,\beta}$ for $c_{S_m,\beta}$.

Note from (2.3) that

$$\lim_{m\to\infty} c_{m,\beta} \frac{1}{n!} \chi^\beta(e) D_m^n = 1, \tag{4.1}$$

where $e \in \mathfrak{S}_n$ is the identity permutation (which is the only permutation with n cycles – all other permutations have fewer cycles).

It follows from (4.1) and the assumption of the proposition that

$$\lim_{m\to\infty} \int \cdots \int \left| c_{m,\beta} \chi^\beta(e) \prod_{i=1}^n K_m(x_i, x_i) - n! \prod_{i=1}^n \kappa(x_i) \right| \mu(dx_1) \dots \mu(dx_n) = 0.$$

To complete the proof, it suffices by (4.1) to show for any permutation $\sigma \neq e$ that $D_m^{-n} \prod_{i=1}^n K_m(x_i, x_{\sigma(i)})$ converges to 0 in $L^1(\mu)$ as $m \to \infty$.

By (1.1), (1.6), and (1.7),

$$\begin{aligned}
\iint \left| D_m^{-1} K_m(x,y) \right|^2 \mu(dx)\mu(dy) &= D_m^{-2} \iint K_m(x,y) K_m(y,x)\, \mu(dx)\mu(dy) \\
&= D_m^{-2} \int K_m(x,x)\, \mu(dx) \\
&= D_m^{-1};
\end{aligned}$$

and, in particular, $D_m^{-1} K_m$ converges to 0 in $\mu^{\otimes 2}$–measure as $m \to \infty$.

Therefore, for $k \geq 2$, $D_m^{-k} \prod_{i=1}^k K_m(x_i, x_{i+1})$ (with the indices defined modulo k so that $k+1 = 1$) converges to 0 in $\mu^{\otimes 2}$–measure as $m \to \infty$. Moreover, by the Cauchy–Schwarz inequality (cf. (1.4)),

$$D_m^{-k} \left| \prod_{i=1}^k K_m(x_i, x_{i+1}) \right| \leq D_m^{-k} \prod_{i=1}^k K_m(x_i, x_i);$$

and, by assumption, the right–hand side converges in $L^1(\mu)$ as $m \to \infty$. Hence, by dominated convergence,

$$\lim_{m\to\infty} D_m^{-k} \int \cdots \int \left| \prod_{i=1}^{k} K_m(x_i, x_{i+1}) \right| \mu(dx_1)\ldots\mu(dx_k) = 0.$$

For $\sigma \neq e$, factor the multiple integral

$$\int \cdots \int \left| D_m^{-n} \prod_{i=1}^{n} K_m(x_i, x_{\sigma(i)}) \right| \mu(dx_1)\ldots\mu(dx_n)$$

into a product of multiple integrals, with one term for each cycle of σ. It is clear from the above that the terms corresponding to k–cycles with $k \geq 2$ (of which there is at least one) converge to 0, whereas the terms corresponding to 1–cycles converge to 1 by assumption. □

Example 4.2. Suppose that Σ is a compact group with Haar measure μ (normalised to be a probability measures). Consider any infinite sequence $\{U^{(k)}\}_{k\in\mathbb{N}}$ of (inequivalent) irreducible unitary representations of Σ. By the Peter–Weyl theorem, each $U^{(k)}$ is finite–dimensional with dimension we will denote by d_k. Let $U_{ij}^{(k)}(x)$, $1 \leq i, j \leq d_k$, $x \in \Sigma$, denote the entries in a matrix realisation of $U^{(k)}$. The functions $\{\sqrt{d_k} U_{ij}^{(k)} : 1 \leq i, j, \leq d_k,\ k \in \mathbb{N}\}$ are orthonormal in $L^2(\mu)$. Let S_m denote the space spanned by $\{\sqrt{d_k} U_{ij}^{(k)} : 1 \leq i, j, \leq d_k,\ 1 \leq k \leq m\}$. Note that

$$\begin{aligned} K_{S_m}(x,x) &= \sum_{k=1}^{m} d_k \sum_{i=1}^{d_k} \sum_{j=1}^{d_k} \left| U_{ij}^{(k)}(x) \right|^2 \\ &= \sum_{k=1}^{m} d_k \operatorname{trace} \left[\left(U^{(k)}(x) \right)^* \left((U^{(k)}(x) \right) \right] \\ &= \sum_{k=1}^{m} d_k^2 \\ &= D_{S_m}, \end{aligned}$$

and so the conditions of Proposition 4.1 hold with $\kappa \equiv 1$.

References

[Bar90] A.I. Barvinok. Complexity of computation of immanants, and representations of the general linear group. *Functional Anal. Appl.*, 24:144–145, 1990.

[Bar97] A. Barvinok. Computing mixed discriminants, mixed volumes, and permanents. *Discrete Comput. Geom.*, 18:205–237, 1997.

[Bar99] A. Barvinok. Polynomial time algorithms to approximate permanents and mixed discriminants within a simply exponential factor. *Random Structures Algorithms*, 14:29–61, 1999.

[Cla96] M. Clausen. Algebraische Komplexitätstheorie. III. *Sém. Lothar. Combin.*, 36:Art. S36c, approx. 16 pp. (electronic), 1996.

[DVJ88] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes.* Springer Series in Statistics. Springer–Verlag, New York – Berlin, 1988.

[FH91] W. Fulton and J. Harris. *Representation Theory: a first course.* Number 129 in Graduate Texts in Mathematics. Springer–Verlag, New York, 1991.

[GM84] R. Grone and R. Merris. An algorithm for the second immanant. *Math. Comp.*, 43:589–591, 1984.

[Har85] W. Hartmann. On the complexity of immanants. *Linear and Multilinear Algebra*, 18:127–140, 1985.

[Hey88] P. Heyfron. Immanant dominance orderings for hook partitions. *Linear and Multilinear Algebra*, 24:65–78, 1988.

[Jam87] G. James. Permanents, immanants, and determinants. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47, Part 2 of *Proc. Sympos. Pure Math.*, pages 431–436. Amer. Math. Soc., Providence, RI, 1987.

[Jam92] G. James. Immanants. *Linear and Multilinear Algebra*, 32:197–210, 1992.

[Lie66] E.H. Lieb. Proofs of some conjectures on permanents. *J. Math. Mech.*, 16:127–134, 1966.

[Lit58] D.E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford, Clarendon Press, 2nd edition, 1958.

[Mac75] O. Macchi. The coincidence approach to stochastic point processes. *Advances in Appl. Probability*, 7:83–122, 1975.

[Meh91] M.L. Mehta. *Random Matrices*. Academic Press, Boston, MA, second edition, 1991.

[Mer87] R. Merris. The permanental dominance conjecture. In R. Grone and F. Uhlig, editors, *Current trends in matrix theory (Auburn, Ala., 1986)*, pages 213–223. North–Holland, New York, 1987.

[Pat92] T.H. Pate. Immanant inequalities and partition node diagrams. *J. London Math. Soc. (2)*, 46:65–80, 1992.

[Pat94] T.H. Pate. Inequalities involving immanants. *Linear Algebra Appl.*, 212/213:31–44, 1994.

[Pat98] T.H. Pate. Row appending maps, Ψ-functions, and immanant inequalities for Hermitian positive semi-definite matrices. *Proc. London Math. Soc. (3)*, 76:307–358, 1998.

[Pat99] T.H. Pate. Tensor inequalities, ξ-functions and inequalities involving immanants. *Linear Algebra Appl.*, 295:31–59, 1999.

[Sch18] I. Schur. Über endliche Gruppen und Hermitische Formen. *Math. Z.*, 1:184–207, 1918.

[Sim96] B. Simon. *Representations of Finite and Compact Groups*, volume 10 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, 1996.

[Sos00] A. Soshnikov. Determinantal random point fields. Preprint available at `http://front.math.ucdavis.edu/math.PR/0002099`, 2000.

[Ste91] J.R. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, 23:422–428, 1991.

[Val79] L.G. Valiant. The complexity of computing the permanent. *Theoret. Comput. Sci.*, 8:189–201, 1979.

E-mail address: `diaconis@math.Stanford.EDU`

Department of Mathematics, Stanford University, Building 380, MC 2125, Stanford, CA 94305, U.S.A.

E-mail address: `evans@stat.Berkeley.EDU`

Department of Statistics #3860, University of California at Berkeley, 367 Evans Hall, Berkeley, CA 94720-3860, U.S.A