LOCAL FIELD U–STATISTICS

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ABSTRACT. Using the classical theory of symmetric functions, a general distributional limit theorem is established for U–statistics constructed from a sequence of independent, identically distributed random variables taking values in a local field with zero characteristic.

1. INTRODUCTION

Since the work of Hoeffding and Halmos in the 1940s, U–statistics constructed from sequences of independent, identically distributed real random variables have played a central role in theoretical and applied statistics. They have also attracted considerable attention from probabilists because they exhibit a rich limit theory that parallels that of i.i.d. sequences (for example, strong laws, central limit theorems, large deviation results, and Berry–Esseen–type theorems have been established for them). Surveys with extensive bibliographies may be found in [Ser80, KB94, Lee90].

Our aim in this paper is to initiate an investigation into the properties of U–statistics on algebraic structures other than the reals: namely, local fields. A local field $\mathbb{K}$ is any locally compact, non-discrete field other than the field of real numbers or the field of complex numbers. All local fields are totally disconnected, and are either finite algebraic extensions of the field of p-adic numbers – in which case the characteristic is zero – or finite algebraic extensions of the the less familiar p-series field (the field of formal Laurent series with coefficients drawn from the finite field with p elements) – in which case the characteristic is non–zero. We give an overview of some of the basic properties of local fields in §2.

Probability on local fields has a substantial, if somewhat scattered, literature, and a comprehensive book–length treatment has yet to be written. For the convenience and interest of the reader we have included a representative (but by no means complete) bibliography in the references.

The natural definition of the notion of U–statistic on the local field $\mathbb{K}$ is the following direct translation of the familiar Euclidean definition.

Definition 1.1. Let $\{X_k\}_{k\in\mathbb{N}}$ be an infinite sequence of independent, identically distributed random variables taking values in the local field $\mathbb{K}$. Fix a symmetric Borel function $\Psi : \mathbb{K}^m \to \mathbb{K}$ for some $m \in \mathbb{N}$ (that is, the value of the function Ψ is unchanged by permutations of its arguments). The sequence of *U–statistics* corresponding to $\{X_k\}_{k\in\mathbb{N}}$ and Ψ is the sequence of $\mathbb{K}$–valued random variables

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$\{Z_k\}_{k \geq m}$ given by

$$Z_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} \Psi(X_{i_1}, X_{i_2}, \ldots, X_{i_m}).$$

The following result is proved in §3. Some remarks on the hypotheses are given after the proof.

Theorem 1.2. *Suppose that the local field $\mathbb{K}$ has characteristic zero, the support of the common distribution of the random variables X_k is compact, and the function Ψ is continuous. Let $\{k(h)\}_{h \in \mathbb{N}}$ be a sequence of positive integers such that $k(h)$ converges to infinity as $h \to \infty$ and also $k(h)$ thought of as an element of $\mathbb{K}$ converges to some $k^* \in \mathbb{K}$ as $h \to \infty$. Then the sequence $\{Z_{k(h)}\}_{h \in \mathbb{N}}$ of U–statistics converges in distribution as $h \to \infty$.*

2. Local fields

This section is essentially a summary of selected results from [Tai75, Sch84]. We refer the reader to these works for a fuller account. Before giving the general definition of a local field, we begin with the prototypical example.

Example 2.1. Fix a positive prime p. We can write any non-zero rational number $r \in \mathbb{Q}\backslash\{0\}$ uniquely as $r = p^s(a/b)$ where a and b are not divisible by p. Set $|r| = p^{-s}$. If we set $|0| = 0$, then the map $|\cdot|$ has the properties:

$$\begin{aligned} |x| = 0 &\Leftrightarrow x = 0; \\ |xy| &= |x||y|; \\ |x+y| &\leq |x| \vee |y|. \end{aligned} \tag{2.1}$$

The map $(x, y) \mapsto |x - y|$ defines a metric on $\mathbb{Q}$ and we denote the completion of $\mathbb{Q}$ in this metric by $\mathbb{Q}_p$. The field operations on $\mathbb{Q}$ extend continuously to make $\mathbb{Q}_p$ a topological field called the *p-adic numbers*. The map $|\cdot|$ also extends continuously and the extension continues to have properties (2.1). The closed unit ball around 0, $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x| \leq 1\}$, is the closure in $\mathbb{Q}_p$ of the integers $\mathbb{Z}$, and is thus a ring (this is also apparent from the properties (2.1)) called the *p–adic integers*. As $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| < p\}$, the set $\mathbb{Z}_p$ is also open. Any other ball around 0 is of the form $\{x \in \mathbb{Q}_p : |x| \leq p^{-k}\} = p^k\mathbb{Z}_p$ for some integer k. Such a ball is the closure of the rational numbers divisible by p^k, and is thus a $\mathbb{Z}_p$–module (this is again also apparent from the properties (2.1)). In particular, such a ball is an additive subgroup of $\mathbb{Q}_p$. Arbitrary balls are translates (= cosets) of these closed and open subgroups. In particular, the topology of $\mathbb{Q}_p$ has a base of closed and open sets, and hence $\mathbb{Q}_p$ is totally disconnected. Further, each of these balls is compact, and hence $\mathbb{Q}_p$ is also locally compact.

A *local field* is a locally compact, non–discrete, totally disconnected, topological field. (As an aside, a locally compact, non-discrete, topological field that is not totally disconnected is necessarily either the real or the complex numbers. A local field with characteristic zero is a finite algebraic extension of the p–adic number field for some prime p. A local field with non–zero characteristic is a finite algebraic extension of the *p–series field*; that is, the field of formal Laurent series with coefficients drawn from the finite field with p elements for some prime p.)

From now on, let $\mathbb{K}$ be a fixed local field. There is a real–valued mapping on $\mathbb{K}$ which we denote by $x \mapsto |x|$. This map has the properties (2.1) and it takes the values $\{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q = p^c$ for some prime p and positive integer c (so that for $\mathbb{K} = \mathbb{Q}_p$ we have $c = 1$).

A map with properties (2.1) is called a *non-archimedean valuation.* The property $|x + y| \le |x| \vee |y|$ is known as the *ultrametric inequality* or the *strong triangle inequality.* The mapping $(x, y) \mapsto |x - y|$ on $\mathbb{K} \times \mathbb{K}$ is a metric for $\mathbb{K}$ that gives the topology of $\mathbb{K}$. A consequence of (2.1) is that if $|x| \ne |y|$, then $|x + y| = |x| \vee |y|$. This latter result implies that for every "triangle" $\{x, y, z\} \subset \mathbb{K}$ we have that at least two of the lengths $|x - y|$, $|x - z|$, $|y - z|$ must be equal and is therefore called the *isosceles triangle property.*

3. Proof of Theorem 1.2

Write E for the support of the common distribution of the X_k. By the ultrametric Stone–Weierstrass theorem (see, for example, §A.4 of [Sch84]), polynomials are uniformly dense in the space of continuous functions from the compact set E^m into $\mathbb{K}$. Therefore, for each $\epsilon > 0$ there exists a polynomial Q such that

$$\sup_{(x_1,\ldots,x_m)\in E^m} |Q(x_1, \ldots, x_m) - \Psi(x_1, \ldots, x_m)| < \epsilon.$$

Define a symmetric polynomial $\overline{Q} : E^m \to \mathbb{K}$ by

$$\overline{Q}(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma\in\mathcal{S}_m} Q(x_{\sigma(1)}, \ldots, x_{\sigma(m)}),$$

where $\mathcal{S}_m$ denotes the symmetric group on m letters and we have used the assumption that $\mathbb{K}$ has zero characteristic to conclude that $m! \ne 0$. By the strong triangle inequality and the symmetry of Ψ,

$$\begin{aligned}
&\sup_{(x_1,\ldots,x_m)\in E^m} \left|\overline{Q}(x_1, \ldots, x_m) - \Psi(x_1, \ldots, x_m)\right| \\
&= \sup_{(x_1,\ldots,x_m)\in E^m} \left| \frac{1}{m!} \sum_{\sigma\in\mathcal{S}_m} \left[Q(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) - \Psi(x_{\sigma(1)}, \ldots, x_{\sigma(m)})\right] \right| \\
&\le |m!|^{-1}\epsilon.
\end{aligned}$$

Thus, again by the strong triangle inequality,

$$\left| \sum_{1\le i_1<i_2<\cdots<i_m\le k} \overline{Q}(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) - Z_k \right| \le |m!|^{-1}\epsilon.$$

It thus clearly suffices to consider the special case of the theorem when Ψ is a symmetric polynomial. By replacing X_k by $X_k - c$ and $\Psi(x_1, \ldots, x_m)$ by $\Psi(x_1 + c, \ldots, x_m + c)$, we may further suppose that $0 \in E$. Moreover, because

$$\lim_h \sum_{1\le i_1<i_2<\cdots<i_m\le k(h)} 1 = \lim_h \binom{k(h)}{m} = \frac{k^*(k^* - 1)\ldots(k^* - m + 1)}{m!}$$

by assumption, we may suppose that Ψ has no constant term.

Given a positive integer n and integers $\lambda_1 \ge \lambda_2 \ge \ldots \ge 0$ with $0 = \lambda_{n+1} = \lambda_{n+2} = \ldots$, define the corresponding *monomial symmetric function* $\mathbf{M}_{n,\lambda}$ in the variables $(x_1, \ldots, x_n)$ by

$$\mathbf{M}_{n,\lambda}(x_1, \ldots, x_n) := \sum_\alpha x^\alpha,$$

where the sum is over all distinct permutations $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $(\lambda_1, \ldots, \lambda_n)$ and

$$x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

The symmetric polynomials $\mathbf{M}_{n,\lambda}$ with $\sum_i \lambda_i \le d$ form a basis for the vector space of symmetric polynomials in $(x_1, \ldots, x_n)$ of total degree at most d (cf. Ch I of [Mac95] or Ch 7 of [Sta99]). Consequently, we have

$$\Psi(x_1, \ldots, x_m) = \sum_\lambda c_\lambda \mathbf{M}_{m,\lambda}(x_1, \ldots, x_m),$$

for suitable constants c_λ, where the sum is over all λ with $0 = \lambda_{m+1} = \lambda_{m+2} = \ldots$, only finitely many of the c_λ are non–zero, and $c_0 = 0$ (by our added assumption that Ψ has no constant term). Observe for such λ that if $\ell := \max\{r : \lambda_r > 0\}$, then

$$\sum_{1 \le i_1 < i_2 < \cdots < i_m \le k} \mathbf{M}_{m,\lambda}(X_{i_1}, \ldots, X_{i_m}) = \binom{k-\ell}{m-\ell} \mathbf{M}_{k,\lambda}(X_1, \ldots, X_k).$$

Note that $\lim_h \binom{k(h)-\ell}{m-\ell}$ exists. It therefore suffices to show that the random vectors

$$(\mathbf{M}_{k,\lambda}(X_1, \ldots, X_k)),$$

where λ ranges over those non–zero partitions such that $0 = \lambda_{m+1} = \lambda_{m+2} = \ldots$ and $\sum_i \lambda_i$ is at most the total degree of Ψ, converge in distribution as $k \to \infty$.

Given a non-negative integer j, define the *power sum symmetric function* $\mathbf{P}_{n,j}$ in the variables $(x_1, \ldots, x_n)$ by

$$\mathbf{P}_{n,j}(x_1, \ldots, x_n) := x_1^j + \cdots + x_n^j,$$

and given integers $\mu_1 \ge \mu_2 \ge \ldots \ge 0$ with $0 = \mu_{n+1} = \mu_{n+2} = \ldots$, set

$$\mathbf{P}_{n,\mu}(x_1, \ldots, x_n) := \prod_i \mathbf{P}_{n,\mu_i}(x_1, \ldots, x_n).$$

We have

$$\mathbf{M}_{n,\lambda}(x_1, \ldots, x_n) = \sum_\mu c_{\lambda\mu} \mathbf{P}_{n,\mu}(x_1, \ldots, x_n)$$

where the sum is over all μ with $\sum_i \mu_i = \sum_i \lambda_i$ and, importantly, the constants $c_{\lambda\mu}$ do not depend on n (cf. Ch I of [Mac95] or Ch 7 of [Sta99], and note that we are again using the assumption that $\mathbb{K}$ has characteristic 0).

It thus suffices to show for any positive integer J that the random vectors

$$\left(\sum_{i=1}^k X_i, \sum_{i=1}^k X_i^2, \ldots, \sum_{i=1}^k X_i^J\right)$$

converge in distribution as $k \to \infty$. However, this process is just a random walk on the compact subgroup of the additive group of $\mathbb{K}^J$ generated by $\{(x, x^2, \ldots, x^J) : x \in E\}$, and the added assumption that $0 \in E$ ensures that the random walk converges in distribution to Haar measure on this subgroup. □

Remark 3.1. The hypothesis that $\mathbb{K}$ has zero characteristic was used several times in the above proof. We do not know whether the result has a counterpart for non–zero characteristics.

Remark 3.2. The role played by the hypothesis that $k(h)$ converges in $\mathbb{K}$ is apparent from the proof. However, because the assumption initially looks rather unusual, we emphasise its importance with the following simple example. Suppose that $\mathbb{K}$ is the field of p–adic numbers $\mathbb{Q}_p$, $m = 2$, and $\Psi(x_1, x_2) = x_1 + x_2$. Then

$$Z_k = (k-1) \sum_{i=1}^{k} X_i.$$

As we observed in the proof, if 0 is in the support of the common distribution of X_k, then $\sum_{i=1}^{k} X_i$ converges in distribution as $k \to \infty$ to Haar measure on the subgroup of $\mathbb{Q}_p$ generated by the support. Note that if k is of the form $p^s + 1$, then $|k-1| = p^{-s}$, whereas if k is of the form $p^s + 2$, $s \geq 1$, then p does not divide $k-1$ and hence $|k-1| = 1$. Consequently, we must take $k \to \infty$ along a subsequence in order for Z_k to have a limit in distribution.

Remark 3.3. In principle, the steps in the proof can be reversed to describe the limiting distribution as the push–forward by an appropriate function of Haar measure on the compact additive subgroup of $\mathbb{K}^{\mathbb{N}}$ generated by $(x, x^2, x^3, \ldots)$ for x in suitable fixed translate of the support of the distribution of the X_k. It does not appear that one can give a more effective characterisation of the limit. In the next section we examine some particularly simple examples where it possible to say something concrete about the limit.

4. Some examples

Suppose that $\mathbb{K}$ is the field of p–adic numbers $\mathbb{Q}_p$ for some prime p and X_k takes only the values 0 and 1 with positive probability. Then $\sum_{i=1}^{k} X_i = \sum_{i=1}^{k} X_i^2 = \ldots$, and these sums converge in distribution to Haar measure on the ring of p–adic integers $\mathbb{Z}_p$.

Write $\mathbf{E}_{n,r}$ for the r^{th} *elementary symmetric function* of n variables; that is,

$$\mathbf{E}_{n,r}(x_1, \ldots, x_n) := \sum_{1 \leq i_1 < \ldots < i_r \leq n} x_{i_1} \ldots x_{i_r}, \quad n \geq r.$$

If we put $\Psi = \mathbf{E}_{m,r}$, then $Z_k = \binom{k-r}{m-r} \mathbf{E}_{k,r}(X_1, \ldots, X_k)$. Write U for a random variable with Haar distribution on $\mathbb{Z}_p$. Then, by a classical determinantal identity (see Example 8 in §I.2 of [Mac95]), $\mathbf{E}_{k,r}(X_1, \ldots, X_k)$ converges in distribution as $k \to \infty$ to

$$\frac{1}{r!} \det \begin{pmatrix} U & 1 & 0 & \ldots & 0 & 0 \\ U & U & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ U & U & U & \ldots & U & r-1 \\ U & U & U & \ldots & U & U \end{pmatrix} = \frac{U(U-1)\ldots(U-r+1)}{r!} = \binom{U}{r}$$

(this is also clear from elementary considerations: if $\{x_1, \ldots, x_n\} \subseteq \{0, 1\}$, then $\mathbf{E}_{n,r}(x_1, \ldots, x_n)$ counts how many subsets of size r can be drawn from a set of $(x_1 + \cdots + x_n)$ objects).

We note in passing that the random variable U is the simplest example of the natural analogue on $\mathbb{Q}_p$ of a Gaussian random variable. Moreover, the random variables $\binom{U}{r}$ are, in a very natural sense, orthogonal and appear in a theory of stochastic integration and Wiener chaos on $\mathbb{Q}_p$. We refer the reader to [Eva89a, Eva91, Eva93, Eva95] for details.

Write $\mathbf{H}_{n,r}$ for the r^{th} *complete symmetric function* of n variables; that is, $\mathbf{H}_{n,r}(x_1, \ldots, x_n)$ is the sum of all monomials of total degree r in the variables $x_1, \ldots, x_n$. By another classical determinantal identity (see Example 8 in §I.2 of [Mac95]), $\mathbf{H}_{k,r}(X_1, \ldots, X_k)$ converges in distribution as $k \to \infty$ to

$$\frac{1}{r!} \det \begin{pmatrix} U & -1 & 0 & \ldots & 0 & 0 \\ U & U & -2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ U & U & U & \ldots & U & -(r-1) \\ U & U & U & \ldots & U & U \end{pmatrix} = (-1)^r \binom{-U}{r}.$$

Note that $\{\mathbf{H}_{k,r}(X_1, \ldots, X_k)\}$ is not a sequence of U–statistics for some function Ψ.

Finally, given a partition λ of some integer r, let $\mathbf{S}_{n,\lambda}(x_1, \ldots, x_n)$ be the *Schur function* in the variables $x_1, \ldots, x_n$ associated with λ (see §I.3 of [Mac95]). From a classical determinantal formula expressing Schur functions in terms of the complete symmetric functions (see Equation (3.4) of [Mac95])

$$\mathbf{S}_{k,\lambda}(X_1, \ldots, X_k) = \det \left(\mathbf{H}_{k,\lambda_i - i + j}(X_1, \ldots, X_k)\right)_{1 \le i,j \le N}$$

for any N that is at least the length of the partition λ (that is, the number of non–zero parts in λ). By the above, the right–hand side converges in distribution as $k \to \infty$ to

$$\det \left((-1)^{\lambda_i - i + j} \binom{-U}{\lambda_i - i + j} \right)_{1 \le i,j \le N} = (-1)^r \det \left(\binom{-U}{\lambda_i - i + j} \right)_{1 \le i,j \le N}.$$

From Example 4 in §I.3 of [Mac95], we see that the right–hand side is

$$\prod_{x \in \lambda'} \frac{U - c_{\lambda'}(x)}{h_{\lambda'}(x)},$$

where λ' is the partition dual to λ and $c_{\lambda'}(x)$ (resp. $h_{\lambda'}(x)$) denotes the *content* (resp. *hook length*) of λ' at x (that is, if x is the box in the i^{th} row and j^{th} column of the Young diagram of λ', then $c_{\lambda'}(x) := j - i$ and $h_{\lambda'}(x)$ is the number of boxes in the hook with corner at x). There is a natural identification of boxes in the Young diagram of a partition with boxes in the Young diagram of the dual partition. Under this identification, $c_{\lambda'}(x) = -c_\lambda(x)$ and $h_{\lambda'}(x) = h_\lambda(x)$. Consequently, $\mathbf{S}_{k,\lambda}(X_1, \ldots, X_k)$ converges in distribution as $k \to \infty$ to

$$\prod_{x \in \lambda} \frac{U + c_\lambda(x)}{h_\lambda(x)}.$$

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