

# UBIQUITY OF SYNONYMY: ALMOST ALL LARGE BINARY TREES ARE NOT UNIQUELY IDENTIFIED BY THEIR SPECTRA OR THEIR IMMANANTAL POLYNOMIALS

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ABSTRACT. There are several common ways to encode a tree as a matrix, such as the adjacency matrix, the Laplacian matrix (that is, the infinitesimal generator of the natural random walk), and the matrix of pairwise distances between leaves. Such representations involve a specific labeling of the vertices or at least the leaves, and so it is natural to attempt to identify trees by some feature of the associated matrices that is invariant under relabeling. An obvious candidate is the spectrum of eigenvalues (or, equivalently, the characteristic polynomial). We show for any of these choices of matrix that the fraction of binary trees with a unique spectrum goes to zero as the number of leaves goes to infinity. We investigate the rate of convergence of the above fraction to zero using numerical methods. For the adjacency and Laplacian matrices, we show that that the *a priori* more informative immanantal polynomials have no greater power to distinguish between trees.

## 1. INTRODUCTION

Tree shape theory furnishes numerical statistics about the structure of a tree [Fel04, MH97]. (Because we are interested in applications of tree statistics to trees that describe evolutionary histories, we will, for convenience, always take the term *tree* without any qualifiers to mean a **rooted, binary tree without any labeling of the vertices**.) Such statistics have two related uses. Firstly, they can be used in an attempt to tell whether two trees are actually the same and, secondly, they can be used to indicate the degree of similarity between two trees with respect to some criterion.

Examples of the latter use are the testing of hypotheses about macroevolutionary processes and the detection of bias in phylogenetic reconstruction. Historically, numerical statistics for such purposes have attempted to capture the notion of the *balance* of a tree, which is the degree to which daughter subtrees are the same size. The balance is typically measured by ad-hoc formulae that are often selected for statistical power to distinguish between two different distributions on trees [KS93, AP02].

In this paper we take some steps in investigating the possibility of a more mathematically “canonical” algebraic approach to tree statistics based on various matrix representations of the tree. Our focus is on the use of statistics for distinguishing trees rather than quantifying degrees of similarity/dissimilarity.

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We first describe the matrix representations of a tree that we will consider.

In algebraic graph theory [Big93], the basic matrix associated to a graph is the *adjacency matrix*  $A(G)$ , whose  $ij^{\text{th}}$  entry is one if  $i$  and  $j$  are connected by an edge, and zero otherwise. From a probabilistic point of view, the more natural matrix to associate with a graph is the *Laplacian matrix*  $L(G)$ , which is the infinitesimal generator of the natural random walk on the graph and is given by  $D(G) - A(G)$ , where  $D(G)$  is the diagonal matrix of vertex degrees. It is clear that a graph can be recovered from either its adjacency or Laplacian matrix. Some authors, such as [Chu97], define the Laplacian to be  $D(G)^{-1/2}L(G)D(G)^{1/2}$ . Note that this difference is not relevant if one is only considering characteristics of the matrix  $L$  such as eigenvalues that are invariant under similarity transformations.

Readers familiar with the phylogenetics literature may be more familiar with the *pairwise distance matrix* [Fel04, SS03]. The distance matrix  $P$  given a leaf-labeling  $1, \dots, n$  has as its  $ij^{\text{th}}$  entry the length of the path between leaf  $i$  and leaf  $j$ . Any leaf-labeled tree is uniquely determined by its distance matrix. These matrices have also been extensively studied as discrete metric spaces [Gup00, Mat02].

The definition of the adjacency and Laplacian matrices requires a numbering of the vertices, while the definition of the distance matrix requires a numbering of the leaves. Because we are considering unlabeled trees (that is, we identify trees that are equivalent in the usual sense of graph-theoretic isomorphism), we are only interested in tree statistics that are invariant under renumbering. Algebraically, this means that we are only interested in features of the associated matrix that are unaffected by similarity transformations via a permutation matrix. The most obvious such statistics are the eigenvalues.

The adjacency and Laplacian matrices and their eigenvalues are familiar objects in the area of spectral graph theory [Big93, Chu97, CDS95]. The eigenvalues of the adjacency matrix tend to contain combinatorial information about the graph, such as bounds on the chromatic number. The eigenvalues of the Laplacian give information of a more geometric flavor, such as the equivalent of the surface area to volume ratio of subgraphs of a graph. As well as having connections to the theory of random walks on graphs, the Laplacian eigenvalues can be used to define the expander graphs, an important class of graphs that have applications in coding theory. Therefore, it would not be too surprising if these eigenvalues were a convenient way to summarize information about a tree, giving a nice collection of tree statistics.

Similarly, it seems plausible that the eigenvalues of the pairwise distance matrix could contain quite a lot of information about the tree that could be used to compare trees. Moreover, although the distance matrix formally contains the same information as the adjacency or Laplacian matrices, the transformation that takes the distance matrix to one of the other two is distinctly non-linear, and hence there is no reason to believe that there is any simple connection between the corresponding eigenvalues.

We demonstrate below that not only do there exist pairs of trees that have the same spectrum as another tree for the adjacency, Laplacian, and distance matrices, but that this is the rule rather than the exception as the trees become large, in the sense that the fraction of trees with a given number of leaves that have a unique adjacency or Laplacian spectrum goes to zero as the number of leaves goes to infinity.

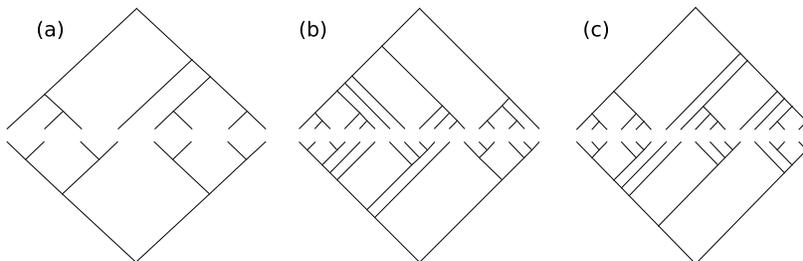


FIGURE 1. Pairs of trees with similar algebraic properties. Figure (a) shows the smallest pair of rooted binary trees with the same adjacency and Laplacian spectrum. Figure (b) shows two trees with the exchange property for both the adjacency and Laplacian matrices. Figure (c) shows two trees with the exchange property for the pairwise distance matrix.

The basic methodology that we use to prove this result was first established in [Sch73] and developed in [BM93] for general (that is, not necessarily bifurcating) graph-theoretic trees in the case of the adjacency and Laplacian matrices. The present paper provides the first results of this type concerning rooted bifurcating trees, as well as the first examination of such results for the pairwise distance matrix. The key idea is to establish that certain pairs of trees  $T_1$  and  $T_2$  have the following *exchange property* for a given matrix representation: exchanging  $T_1$  for  $T_2$  as subtrees of a given tree does not change the spectrum for that matrix representation. It then becomes a matter of showing that the number of trees with a given number of leaves is asymptotically of larger order than the number of trees with the same number of leaves that don't have a particular subtree. For this we build on the generating function argument used in [Wed22] for asymptotic estimates of the number of unlabeled rooted bifurcating trees (see Section 3).

One possible explanation for this phenomenon is that two matrices have the same spectrum if they are similar via an arbitrary similarity transformation rather than just via a permutation transformation, and this suggests considering features of a matrix that are invariant under permutation similarities but not more general ones. We will now describe a feature of a matrix, its *immanantal polynomial*, that has this property.

The *immanant* is a generalization of the determinant. Recall that the determinant of a matrix  $A = (a_{ij})$  is given by

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{i\sigma(i)},$$

where the sum is over the symmetric group of permutations of  $\{1, 2, \dots, n\}$  and  $\operatorname{sgn}(\sigma)$  is the *sign* of the permutation  $\sigma$ .

The function  $\operatorname{sgn}$  is a particular example of a *character* of an *irreducible representation* of the symmetric group. It would take us too far afield to define these notions here, but we note that classical representation theory is one of the gems of pure mathematics and excellent treatments may be found in [Rob61, FH91, Sim96, Sag01]. We note, however, that the irreducible characters are constant on the conjugacy classes of the symmetric group (recall that two permutations belong to

the same conjugacy class if and only if they have the same cycle structure) and they form a basis for the vector space of functions with this property (the *class functions*).

Our use of characters is simply to define the immanant

$$d_\chi(A) := \sum_{\sigma \in S_n} \chi(\sigma) \prod_i a_{i\sigma(i)}$$

of a matrix  $A$  corresponding to the irreducible character  $\chi$ . A discussion of immanants may be found in [LR34, Lit40]. The *immanantal polynomial* of a matrix is the corresponding generalization of the characteristic polynomial; that is, it is the polynomial  $x \mapsto d_\chi(xI - A)$ . Because the characters are class functions, the immanantal polynomial is invariant under similarity by permutation matrices, but it will not typically be invariant under more general similarities.

Unfortunately, as we show in Lemma 2, for either the adjacency or Laplacian matrix the following two conditions on a pair of trees are equivalent:

- the spectra are equal,
- the immanantal polynomials are equal for all irreducible characters.

Consequently, the immanantal polynomials for the adjacency and Laplacian matrices provide no more distinguishing power than the spectra and, in particular, a vanishing fraction of large trees have a unique immanantal polynomial for these matrices. We do not know if the same fact is true for the distance matrix.

Our main result is thus the following.

**Theorem 1.** *Let  $t_n$  be the number of trees with  $n$  leaves. For either the adjacency, Laplacian, or pairwise distance matrix, let  $l_n$  be the number of trees with  $n$  leaves that do not share their spectrum with another tree. Then the fraction  $l_n/t_n$  goes to zero as  $n$  goes to infinity. For the adjacency and Laplacian matrices, the same result holds if we replace the spectrum by the complete set of immanantal polynomials.*

The rate of convergence of the fraction in Theorem 1 is also of interest. If it is extremely slow then the existence of trees with shared spectra may not be practically relevant for the construction of informative tree shape statistics. We investigate this matter numerically in Section 4.

## 2. ALGEBRAIC PRELIMINARIES CONCERNING SPECTRA AND IMMANANTAL POLYNOMIALS

**2.1. Equality of adjacency and Laplacian spectra implies equality of immanantal polynomials.** In order to prove results for the adjacency and Laplacian matrices simultaneously, we define for a tree  $T$  and arbitrary real numbers  $y$  and  $z$  the *generalized Laplacian*  $\tilde{L}(T) := yD(T) + zA(T)$  (recall that  $A(T)$  is the adjacency matrix and  $D(T)$  is the diagonal matrix of vertex degrees). We define the corresponding *generalized Laplacian immanantal polynomial* of the tree  $T$  with  $r$  vertices to be

$$x \mapsto d_\chi \left( xI - \tilde{L}(T) \right)$$

for an irreducible character  $\chi$  of the symmetric group  $S_r$ .

The generalized Laplacian immanantal polynomial can be computed in a simple combinatorial fashion as follows. Define a  *$k$ -matching* to be a set of  $k$  pairwise disjoint edges of the tree (that is, a set of edges such that no two share a common vertex). Let  $M_k(T)$  denote the set of  $k$ -matchings on the tree  $T$ . We think of an

edge as a pair of vertices, so when we use the notation  $i \notin p$  for a vertex  $i$  and an edge  $p$  we mean that  $i$  is not one of the ends of  $p$ . Let  $C_k$  denote the conjugacy class of the symmetric group  $S_r$  consisting of permutations that are the product of  $k$  disjoint cycles, and write  $\chi(C_k)$  for the common value of the character  $\chi$  on such permutations. The following lemma appears in [BM93] and is included for completeness.

**Lemma 1.** *The generalized Laplacian immanantal polynomial of the tree  $T$  for the character  $\chi$  is given by*

$$\sum_{k \geq 0} \chi(C_k) z^{2k} \sum_{p \in M_k(T)} \prod_{i \notin p} (x - y d_i(T)).$$

*Proof.* Set  $M := xI - \tilde{L}(T) = (m_{ij})$  so that the generalized Laplacian immanantal polynomial is

$$(1) \quad \sum_{\sigma \in S_n} \chi(\sigma) \prod_i m_{i\sigma(i)}.$$

The matrix entries  $m_{ij}$  are zero unless  $i = j$  or there is an edge between  $i$  and  $j$ . If the permutation  $\sigma$  has a cycle of length 3 or greater, then corresponding term in (1) must be zero because otherwise the tree would have a loop. Therefore we need only consider permutations that are products of disjoint transpositions where, moreover, each transposition exchanges the two vertices of an edge. Such a permutation is equivalent in an obvious way to a  $k$ -matching for some  $k$ , and the lemma follows.  $\square$

**Lemma 2.** *Two trees have the same spectrum for their generalized Laplacian if and only if they have the same generalized Laplacian immanantal polynomial for all characters.*

*Proof.* One direction is trivial: if two trees have the same generalized Laplacian immanantal polynomial for all characters, then their generalized Laplacians have the same characteristic polynomial and hence the same spectrum.

Conversely, if the generalized Laplacians of two trees have the same spectrum, then the characteristic polynomials of the generalized Laplacians are the same. Lemma 1 in the case  $\chi = \text{sgn}$ , the fact that  $\text{sgn}(C_k) = \pm 1$  for all  $k$ , and the fact that two equal polynomials have the same coefficients imply that the quantity

$$\sum_{p \in M_k(T)} \prod_{i \notin p} (x - y d_i(T))$$

is the same for both trees. Another application of Lemma 1 completes the proof.  $\square$

**2.2. A sufficient condition for two trees to have the same adjacency or Laplacian spectrum.** We use the phylogenetic rather than graph-theoretic definition of a subtree. That is, a subtree of a given rooted tree is what results from separating an edge from its vertex furthest from the root, which then becomes the root of the subtree.

Recall that  $M_k(T)$  is the set of  $k$ -matchings of the tree  $T$ . Let  $N_k(T)$  be the set of  $k$ -matchings where the chosen edges do not contain the root.

Define

$$P_k(T) := \sum_{p \in M_k(T)} \prod_{i \notin p} (x - yd_i(T))$$

$$Q_k(T) := \sum_{p \in N_k(T)} \prod_{i \notin p} (x - yd_i(T))$$

The following lemma is implicit in [BM93], but again we include a proof for completeness.

**Lemma 3.** *Let  $S_1$  and  $S_2$  be trees with the same number of leaves. If  $P_k(S_1) = P_k(S_2)$  and  $Q_k(S_1) = Q_k(S_2)$  for all  $k$ , then any tree with  $S_1$  as a subtree has the same generalized Laplacian spectrum as the tree obtained by substituting  $S_2$  for  $S_1$ .*

*Proof.* Let  $T_1$  be a tree with  $S_1$  as a subtree, and write  $T_2$  for the tree obtained by substituting  $S_2$  for  $S_1$ . Denote by  $e_0$  the edge that connects the rest of  $T_1$  (resp.  $T_2$ ) to the root of  $S_1$  (resp.  $S_2$ ).

We differentiate between two types of  $k$ -matchings of  $T_i$ : those that contain  $e_0$  and those that do not. Note that a  $k$ -matching of  $T_i$  that **does not** contain  $e_0$  restricts to an  $\ell$ -matching of  $S_i$  for some  $\ell$ , and all matchings of  $S_i$  arise via such a restriction. Similarly, a  $k$ -matching of  $T_i$  that **does** contain  $e_0$  restricts to an  $\ell$ -matching of  $S_i$  with the property that the root of  $S_i$  does not belong to any edge in the matching, and all matchings of  $S_i$  with this property arise via such a restriction.

Consider the formula for the characteristic polynomial of the generalized Laplacian matrix that comes from Lemma 1 with  $\chi = \text{sgn}$ . Apply this formula to  $T_1$  and  $T_2$ . The assumption  $P_k(S_1) = P_k(S_2)$  (resp.  $Q_k(S_1) = Q_k(S_2)$ ) ensures that the matchings that do not include (resp. do include)  $e_0$  make the same contribution to the respective characteristic polynomials.  $\square$

The trees depicted in Figure 1 (b) are the smallest pair of rooted bifurcating trees satisfying the criteria of this lemma. The verification of this fact was done by computer, and the corresponding  $P_k$  and  $Q_k$  polynomials are available from the authors upon request.

**2.3. A sufficient condition for two trees to have the same distance matrix spectrum.** We first recall an identity for determinants of partitioned matrices. If

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

then

$$\begin{aligned} \det C &= \det \left( \begin{pmatrix} I & -C_{12}C_{22}^{-1} \\ 0 & I \end{pmatrix} C \begin{pmatrix} I & 0 \\ -C_{22}^{-1}C_{21} & I \end{pmatrix} \right) \\ (2) \quad &= \det \begin{pmatrix} C_{11} - C_{12}C_{22}^{-1}C_{21} & 0 \\ 0 & C_{22} \end{pmatrix} \\ &= \det(C_{22}) \det(C_{11} - C_{12}C_{22}^{-1}C_{21}) \\ &= \det(C_{11}) \det(C_{22} - C_{21}C_{11}^{-1}C_{12}). \end{aligned}$$

**Lemma 4.** *Form two trees  $T_1$  and  $T_2$  by gluing trees  $S_1$  and  $S_2$  with distance matrices  $A_1$  and  $A_2$  onto the same leaf of a common tree  $R$ . Write  $a_i$  for the vector*

of distances from the leaves of  $S_i$  to the root of  $S_i$ . Suppose that the following pairs of matrices have the same spectra:

$$A_i, \quad i = 1, 2,$$

$$\begin{pmatrix} A_i & a_i \\ a_i' & 0 \end{pmatrix}, \quad i = 1, 2,$$

$$\begin{pmatrix} A_i & a_i \\ \mathbf{1}' & 0 \end{pmatrix}, \quad i = 1, 2,$$

$$\begin{pmatrix} A_i & \mathbf{1} \\ a_i' & 0 \end{pmatrix}, \quad i = 1, 2,$$

and

$$\begin{pmatrix} A_i & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}, \quad i = 1, 2,$$

where  $\mathbf{1}$  is a column vector with each entry 1. Then the distance matrices of  $T_1$  and  $T_2$  have the same spectrum.

*Proof.* Write  $B$  for the distance matrix of  $R$ . Then  $B$  has the partitioned form

$$\begin{pmatrix} \check{B} & b \\ b' & 0 \end{pmatrix},$$

where  $\check{B}$  is the distance matrix of the tree obtained from  $R$  by deleting the last leaf,  $b$  is the column vector of distances from the other leaves of  $R$  to the last leaf. Assume without loss of generality that this last leaf is the attachment point of the  $S_i$ .

Denote by  $D_i$  the distance matrix of  $T_i$ . Observe that

$$D_i = \begin{pmatrix} \check{B} & b\mathbf{1}' + \mathbf{1}a_i' \\ a_i\mathbf{1}' + \mathbf{1}b' & A_i \end{pmatrix},$$

Hence, by (2),  $D_i$  has the characteristic polynomial

$$\begin{aligned} \det(xI - D_i) &= \det(xI - A_i) \det[(xI - \check{B}) - (-b\mathbf{1}' - \mathbf{1}a_i')(xI - A_i)^{-1}(-a_i\mathbf{1}' - \mathbf{1}b')] \\ &= \det(xI - A_i) \det \left[ (xI - \check{B}) \right. \\ &\quad - (\mathbf{1}'(xI - A_i)^{-1}a_i) b\mathbf{1}' \\ &\quad - (\mathbf{1}'(xI - A_i)^{-1}\mathbf{1}) bb' \\ &\quad - (a_i'(xI - A_i)^{-1}a_i) \mathbf{1}\mathbf{1}' \\ &\quad \left. - (a_i'(xI - A_i)^{-1}\mathbf{1}) \mathbf{1}b' \right]. \end{aligned}$$

Using (2) again, we see that a partitioned matrix of the form

$$\begin{pmatrix} A & g \\ h' & 0 \end{pmatrix},$$

where  $g$  and  $h$  are column vectors, has characteristic polynomial

$$\det(xI - A)[x - h'(xI - A)^{-1}g],$$

and the result follows.  $\square$

## 3. ASYMPTOTIC NUMBERS OF TREES

As outlined in the Introduction, the proof of Theorem 1 follows immediately from Lemma 2, Lemma 3, Lemma 4, and the following result.

**Proposition 1.** *Let  $t_n$  be the number of trees with  $n$  leaves. Let  $s_n$  be the number of such trees that do not contain a given subtree. Then the fraction  $s_n/t_n$  goes to zero as  $n$  goes to infinity.*

*Proof.* Suppose that the forbidden subtree has  $a$  leaves. Let  $f(x) := \sum_{i=1}^{\infty} t_i x^i$  and  $f_a(x) := \sum_{i=1}^{\infty} s_i x^i$  denote the generating functions for  $t_n$  and  $s_n$ , respectively. Write  $\rho$  for the radius of convergence of the power series  $f$  and  $\rho_a$  for the radius of convergence of the power series  $f_a$ . Note that  $\rho \leq \rho_a < 1$ .

It is shown in [Ott48] that  $\rho = 0.402698\dots$  and

$$\lim_{n \rightarrow \infty} n^{3/2} \rho^n t_n = \eta,$$

where  $\eta = 0.7916032\dots$  (see [Lan77] for an asymptotic expansion of  $t_n$  that extends this result and [HRS75, HRS86, Ben74, Ben76] for reviews of general methods for determining asymptotic numbers of trees of various sorts from a knowledge of the functional equations that their generating functions solve). Since  $s_n$  is  $o(\alpha^{-n})$  for any  $\alpha < \rho_a$ , it follows that  $s_n/t_n$  is  $o(\beta^n)$  for any  $\beta > \rho/\rho_a$ , and the proposition will hold if we can show that  $\rho < \rho_a$ .

For the sake of completeness and because it serves as a good introduction to the derivation of the functional equation satisfied by the generating function of  $s_n$ , we first derive the well-known functional equation satisfied by the generating function of the  $t_n$ . See the comments after the proof of the lemma for some remarks about the history of the latter generating function.

By decomposing a tree into the two subtrees rooted at the daughters of the root, it is clear that

$$\begin{aligned} t_n &= t_1 t_{n-1} + t_2 t_{n-2} + \cdots + t_{m+1} t_{m-1}, & \text{for } n = 2m + 1, \\ t_n &= t_1 t_{n-1} + t_2 t_{n-2} + \cdots + t_{m+1} t_{m-1} + t_m (t_m + 1)/2, & \text{for } n = 2m. \end{aligned}$$

These expressions are equivalent to the statement

$$(3) \quad \sum_{i=1}^{n-1} t_i t_{n-i} = 2t_n + t_{n/2}$$

where  $t_{n/2}$  is set to zero unless  $n$  is even.

From (3) the generating function  $f$  satisfies the functional equation

$$\begin{aligned} f^2(x) &= \sum_{n=2}^{\infty} x^n \sum_{i=1}^{n-1} t_i t_{n-i} \\ &= \sum_{n=2}^{\infty} x^n (2t_n - t_{n/2}) \\ &= 2f(x) - f(x^2) - 2x. \end{aligned}$$

It will be convenient to consider the function  $g := 1 - f$ , which satisfies the functional equation

$$(4) \quad g(x^2) = 2x + g^2(x).$$

It is shown in [Wed22] that:

- The radius of convergence  $\rho$  is strictly positive.
- The functional equation (4) has a unique solution in the whole complex plane, and this solution agrees with our power series in  $\{x \in \mathbb{C} : |x| < \rho\}$ .
- If, with a slight abuse of notation, we also denote this solution by  $g$ , then  $g(\rho) = 0$ .
- The point  $\rho$  is the only zero of  $g$  within  $\{x \in \mathbb{C} : |x| < 1\}$ .

It is clear from the power series that  $g$  is continuous and decreasing on  $[0, \rho)$  and  $g(0) = 1$ . Hence  $g$  is strictly positive on  $[0, \rho)$ .

As observed in [Wed22], these observations suggest a method for computing  $\rho$ . Put  $h(x) = g(x)/\sqrt{x}$ , so that  $h$  satisfies  $h(x^2) = 2 + h^2(x)$ . Set

$$w_k(\eta) := \left(2 + \eta^{2^k}\right)^{2^{-k}}, \quad \eta \in \mathbb{R},$$

and

$$q_n := w_{n-1} \circ w_{n-2} \cdots \circ w_0,$$

so that each function  $q_n$  is strictly increasing on  $[-2, \infty)$  and  $q_1 \leq q_2 \leq \dots$ . In particular,  $q_n$  converges pointwise as  $n \rightarrow \infty$ . Moreover,

$$\lim_{n \rightarrow \infty} q_n(h(x)) = \lim_{n \rightarrow \infty} (h(x^{2^n}))^{2^{1-n}} = \lim_{n \rightarrow \infty} \frac{(g(x^{2^n}))^{2^{1-n}}}{x} = \frac{1}{x}$$

for  $0 < x < 1$ . Therefore

$$\frac{1}{\rho} = \lim_{n \rightarrow \infty} q_n(0).$$

Conveniently, (3) holds with  $s_n$  in place of  $t_n$  for all  $n$  except for  $n = a$ ; in this case one simply adds two to the right hand side of the equation to make up for the fact that  $s_a = t_a - 1$ . Hence  $f_a$  satisfies the functional equation.

$$(5) \quad f_a^2(x) = 2f_a(x) - f_a(x^2) - 2x + 2x^a.$$

Set  $g_a := 1 - f_a$ , so that

$$(6) \quad g_a(x^2) = 2x - 2x^a + g_a^2(x).$$

It is clear that  $g_a$  is continuous and decreasing on  $[0, \rho_a)$  and  $g_a(0) = 1$ . Following the arguments in [Wed22], the functional equation (6) has a unique solution in the whole complex plane, and this solution agrees with our power series in  $\{x \in \mathbb{C} : |x| < \rho_a\}$ . Moreover, analogues of the other properties of  $g$  obtained in [Wed22] hold for  $g_a$ .

Set  $h_a(x) = g_a(x)/\sqrt{x}$ , so that

$$h_a(x^2) = 2 - 2x^{a-1} + h_a^2(x).$$

Put

$$w_{k,a,\xi}(\eta) := \left(2 - 2\xi^{2^k} + \eta^{2^k}\right)^{2^{-k}}, \quad \eta \in \mathbb{R},$$

and

$$q_{n,a,\xi} := w_{n-1,a,\xi} \circ w_{n-2,a,\xi} \cdots \circ w_{0,a,\xi}.$$

Then

$$\lim_{n \rightarrow \infty} q_{n,a,\xi^{a-1}}(h(x)) = \lim_{n \rightarrow \infty} (h_a(x^{2^n}))^{2^{1-n}} = \lim_{n \rightarrow \infty} \frac{(g_a(x^{2^n}))^{2^{1-n}}}{x} = \frac{1}{x}$$

for  $0 < x < 1$ , and, in particular,

$$\frac{1}{\rho_a} = \lim_{n \rightarrow \infty} q_{n,a,\rho_a^{a-1}}(0).$$

Now

$$\begin{aligned} q_{n,a,\rho_a^{a-1}}(0) &= w_{n-1,a,\rho_a^{a-1}} \circ w_{n-2,a,\rho_a^{a-1}} \cdots \circ w_{0,a,\rho_a^{a-1}}(0) \\ &\leq w_{n-1} \circ w_{n-2} \cdots \circ w_1 \circ w_{0,a,\rho_a^{a-1}}(0) \\ &= q_n(-2\rho_a^{a-1}), \end{aligned}$$

and so

$$\frac{1}{\rho_a} \leq \lim_{n \rightarrow \infty} q_n(-2\rho_a^{a-1}) \leq \lim_{n \rightarrow \infty} q_n(0) = \frac{1}{\rho}.$$

It therefore suffices to show that the function  $y \mapsto \lim_{n \rightarrow \infty} q_n(y)$  is strictly increasing on  $(-\varepsilon, \infty)$  for some  $0 < \varepsilon < 2$ .

Observe that

$$q'_n = \prod_{k=1}^{n-1} w'_k \circ q_k.$$

For  $k \geq 1$ ,

$$\begin{aligned} w'_k(x) &= x^{2^k-1} \left(2 + x^{2^k}\right)^{2^{-k}-1} \\ &= \left(2x^{-2^k} + 1\right)^{2^{-k}-1}, \end{aligned}$$

so that  $x \mapsto w'_k(x)$  is non-decreasing for  $x > 0$ . For  $y \in (-\varepsilon, \infty)$ ,

$$w'_k \circ q_k(y) \geq w'_k \circ q_1(y) \geq w'_k \circ q_1(-\varepsilon) = w'_k(2 - \varepsilon)$$

and hence

$$\liminf_{n \rightarrow \infty} \inf_{y > -\varepsilon} q'_n(y) \geq \prod_{k=1}^{\infty} w'_k(2 - \varepsilon).$$

Taking  $0 < \varepsilon < 1$ , the proof will be completed by demonstrating for any  $x > 1$  that

$$\prod_{k=1}^{\infty} w'_k(x) > 0.$$

Taking the logarithm gives

$$\begin{aligned} \sum_{k=1}^{\infty} (2^{-k} - 1) \log \left(2x^{-2^k} + 1\right) &> - \sum_{k=1}^{\infty} \log \left(2x^{-2^k} + 1\right), \\ &> - \sum_{k=1}^{\infty} 2x^{-2^k}, \end{aligned}$$

and this series clearly converges by the ratio test.  $\square$

In relation to Proposition 1, we note from [Sch73] that the number of rooted strictly bifurcating trees without a given subtree is asymptotically smaller than the number of all graph-theoretic trees (see also [Lu96] for more about the enumeration of general trees without a given subtree), but this is not enough for our purposes. We needed to show that it is asymptotically smaller than the space of all rooted strictly bifurcating trees. The generating function for  $t_n$  seems first to have been investigated in [Wed22] in connection with enumerating “types of arrangements” in a commutative but non-associative algebra, such as  $a_1(a_2(a_3a_4))$  or  $(a_1a_2)(a_3a_4)$ ; these are identical to rooted bifurcating trees in the “Newick” format [Fel04]. The

leaves	trees	GLS	DS
2	1	1	1
3	1	1	1
4	2	2	2
5	3	3	3
6	6	6	6
7	11	11	11
8	23	22	23
9	46	45	46
10	98	95	98
11	207	203	207
12	451	443	451
13	983	972	983
14	2179	2159	2179
15	4850	4827	4850
16	10905	10870	10905
17	24631	24580	24630
18	56011	55931	56009

TABLE 1. The number of trees, the number of spectra for the generalized Laplacian, and the number of spectra for the distance matrix.

recursion behind the generating function has been re-discovered independently several times such as in [Eth37] – see [OB49] for a discussion and several further references. We remark that numerically iterating the quantity  $q_n$  of the proof converges quickly to the value of  $\rho^{-1}$  calculable by other means [Ott48, Har71]. We also observe that the methods of [Lan77, HRS75, HRS86, Ben74, Ben76] can be used to show, in the notation of the proof, that  $\lim_{n \rightarrow \infty} n^{3/2} \rho_a^n s_n = \eta_a$  for some positive constant  $\eta_a$  and hence  $\lim_{n \rightarrow \infty} (\rho_a/\rho)^n (s_n/t_n) = \eta_a/\eta$ , but we don't pursue this matter here.

#### 4. NUMERICAL EXPERIMENTS

Proposition 1 says nothing about the rate of convergence of the fraction. Here we investigate this rate using computation. The characteristic polynomials for the generalized Laplacian were calculated via a doubly-recursive algorithm to enumerate matchings. The characteristic polynomials for the distance matrices were calculated via the Leverrier-Faddeev algorithm [Hou98]. These algorithms were implemented in `ocaml` [CMP00] and the code is available upon request.

Table 1 clearly shows that the fraction of trees with unique spectra does not go to zero very quickly. Of course we can't compute this fraction for large numbers of leaves, but we can get some idea of the convergence by using the recursion relation corresponding to the generating function (5). We plot the number of trees that do not have one of two subtrees of size seventeen as a subtree. This is an actual fraction that can be used with Proposition 1 in order to prove Theorem 1 for the generalized Laplacian matrix.

Figure 3 shows that this fraction converges extremely slowly, despite the fact that as shown above it is asymptotically equivalent to  $\beta^n$  for some  $0 < \beta < 1$ . It

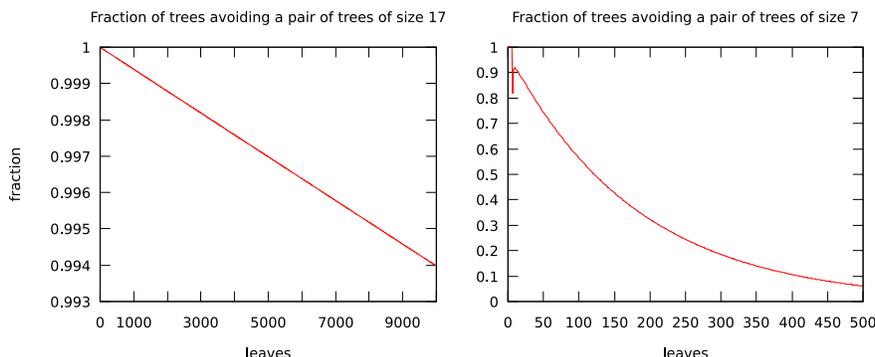


FIGURE 2. The fraction of trees not containing a given pair of subtrees of size 7 and 17.

is important to note, however, that this fraction is probably a very crude upper bound on the fraction of trees that share a spectrum with another tree. To aid comparison we have included the much more quickly converging number of trees without a pair of subtrees of size seven. As can be seen in Table 1, the actual number not sharing a spectrum goes down considerably more quickly, though it is probably still the case that the vast majority of trees of intermediate size should have their own spectra.

In conclusion, we have shown that the fraction of strictly bifurcating rooted trees having a unique spectrum with respect to the adjacency, Laplacian, or pairwise distance matrices goes to zero as the number of leaves goes to infinity. This indicates that the spectra of these matrices cannot perfectly distinguish between trees. However, because the convergence to zero appears to be so slow, these spectra may still provide useful information for tree sizes that appear in practice. We plan to investigate this matter further in a future article.

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