A Zero–one Law for Linear Transformations of Lévy Noise

Steven N. Evans

Abstract. A Lévy noise on $\mathbb{R}^d$ assigns a random real "mass" $\Pi(B)$ to each Borel subset B of $\mathbb{R}^d$ with finite Lebesgue measure. The distribution of $\Pi(B)$ only depends on the Lebesgue measure of B, and if $B_1, \ldots, B_n$ is a finite collection of pairwise disjoint sets, then the random variables $\Pi(B_1), \ldots, \Pi(B_n)$ are independent with $\Pi(B_1 \cup \cdots \cup B_n) = \Pi(B_1) + \cdots + \Pi(B_n)$ almost surely. In particular, the distribution of $\Pi \circ g$ is the same as that of Π when g is a bijective transformation of $\mathbb{R}^d$ that preserves Lebesgue measure. It follows from the Hewitt–Savage zero–one law that any event which is almost surely invariant under the mappings $\Pi \mapsto \Pi \circ g$ for every Lebesgue measure preserving bijection g of $\mathbb{R}^d$ must have probability 0 or 1. We investigate whether certain smaller groups of Lebesgue measure preserving bijections also possess this property. We show that if $d \geq 2$, the Lévy noise is not purely deterministic, and the group consists of linear transformations and is closed, then the invariant events all have probability 0 or 1 if and only if the group is not compact.

1. Introduction

The zero-one law of Hewitt and Savage [**1**] concerns sequences of of independent, identically distributed, random variables $X = \{X_k : k \in \mathbb{Z}\}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It says that if $A \subseteq \mathbb{R}^{\mathbb{Z}}$ is any product measurable set such that gA and A differ by a $\mathbb{P}$-null set for all bijections $g : \mathbb{Z} \to \mathbb{Z}$ that fix all but finitely many elements of $\mathbb{Z}$, then $\mathbb{P}\{X \in A\}$ is either 0 or 1. Of course, it is not important that X is indexed by $\mathbb{Z}$: we could replace $\mathbb{Z}$ by any countable set.

One natural family of continuous analogues of the family of sequences of independent, identically distributed, random variables is the family the Lévy noises. Recall that a Lévy noise on $\mathbb{R}^d$ is defined as follows. Let μ be an infinitely divisible probability measure on $\mathbb{R}$. There is an associated convolution semigroup $(\mu_t)_{t \geq 0}$ of probability measures on $\mathbb{R}$: that is,

- μ_1 is μ
- μ_0 is δ_0, the point mass at 0,

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- $\mu_s * \mu_t = \mu_{s+t}$, for all $s, t \ge 0$, where $*$ denotes convolution,
- the weak limit as $t \downarrow s$ of μ_t is μ_s for all $s \ge 0$.

Denote the Borel σ-field of $\mathbb{R}^d$ by $\mathcal{B}(\mathbb{R}^d)$. Write Λ for Lebesgue measure on $\mathbb{R}^d$ and let $\mathcal{C}(\mathbb{R}^d)$ be the subset of $\mathcal{B}(\mathbb{R}^d)$ consisting of sets with finite Lebesgue measure. A Lévy noise on $\mathbb{R}^d$ corresponding to μ is a collection of real-valued random variables $\Pi = \{\Pi(B) : B \in \mathcal{C}(\mathbb{R}^d)\}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the properties:

- the random variable $\Pi(B)$ has distribution $\mu_{\Lambda(B)}$ for all $B \in \mathcal{C}(\mathbb{R}^d)$,
- if $B_1, \ldots, B_n$ is a finite collection of pairwise disjoint sets in $\mathcal{C}(\mathbb{R}^d)$, then the random variables $\Pi(B_1), \ldots, \Pi(B_n)$ are independent and $\Pi(B_1 \cup \cdots \cup B_n) = \Pi(B_1) + \cdots + \Pi(B_n)$ almost surely.

For each infinitely divisible probability measure μ it is possible to construct (via Kolmogorov's extension theorem) a corresponding Lévy noise on $\mathbb{R}^d$ for every d. Note that if μ is not a point mass, then the random variable $\Pi(B)$ is not almost surely constant when $B \in \mathcal{C}(\mathbb{R}^d)$ is a set with $\Lambda(B) > 0$.

The most familiar examples of Lévy noises are the usual *Gaussian white noise*, in which case μ is the standard Gaussian probability distribution, and the *homogeneous Poisson random measures*, in which case μ is a Poisson distribution with some positive mean.

Let Σ be the Cartesian product $\mathbb{R}^{\mathcal{C}(\mathbb{R}^d)}$ and write $\mathcal{S}$ for the corresponding product σ-field. The Lévy noise Π is a measurable map from $(\Omega, \mathcal{F})$ to $(\Sigma, \mathcal{S})$. Given a bijection $g : \mathbb{R}^d \to \mathbb{R}^d$ that is Borel measurable with a Borel measurable inverse, there is a corresponding bijection $T_g : \Sigma \to \Sigma$ that maps the element $(\pi(B))_{B \in \mathbb{R}^{\mathcal{C}(\mathbb{R}^d)}}$ to the element $(\pi(g^{-1}B))_{B \in \mathbb{R}^{\mathcal{C}(\mathbb{R}^d)}}$. The mapping T_g and its inverse are both measurable. Note that $T_g \circ \Pi$ has the same distribution as Π when g preserves the Lebesgue measure Λ.

If G is a group of Lebesgue measure preserving bijections, then the corresponding *invariant σ-field $\mathcal{I}_G$* is the collection of sets $S \in \mathcal{S}$ with the property

$$\mathbb{P}(\{\Pi \in S\} \triangle \{T_g \Pi \in S\}) = 0 \text{ for all } g \in G,$$

where $\triangle$ denotes the symmetric difference.

It follows readily from the Hewitt–Savage zero–one law that if G is the group of all Borel measurable bijections that have Borel measurable inverses and preserve Lebesgue measure, then the invariant σ-field $\mathcal{I}_G$ consists of events with probability 0 or 1.

However, the same conclusion still holds for much "smaller" groups G. For example, it holds when G is $\mathbb{R}^d$ acting on itself via translations (this follows from the multiparameter ergodic theorem and the Kolmogorov zero–one law). On the other hand, the conclusion fails when μ is not a point mass and G is the group $\mathrm{O}(\mathbb{R}^d)$ of linear transformations of $\mathbb{R}^d$ that preserve the usual Euclidean inner product (equivalently, once we have chosen an orthonormal basis we may think of $\mathrm{O}(\mathbb{R}^d)$ as the group of $d \times d$ orthogonal matrices). Each element of $\mathrm{O}(\mathbb{R}^d)$ preserves Lebesgue measure and the random variable $\Pi(\{x \in \mathbb{R}^d : \|x\| \le 1\})$ is $\mathcal{I}_G$-measurable but not almost surely constant.

Our aim in this paper is to characterize the closed groups of linear transformations of $\mathbb{R}^d$ that preserve Lebesgue measure and for which the corresponding invariant σ-field consists of events with probability 0 or 1.

Recall that a linear mapping of $\mathbb{R}^d$ into itself preserves Lebesgue measure if and only if the corresponding matrix with respect to some basis of $\mathbb{R}^d$ has a determinant with absolute value 1. Of course, if this condition holds for one basis, then it holds for all bases. The collection of linear maps that preserve Lebesgue measure is a group. Denote this group by Γ. We have $\Gamma = (+1) \times \mathrm{Sl}(\mathbb{R}^d) \sqcup (-1) \times \mathrm{Sl}(\mathbb{R}^d)$, where $\mathrm{Sl}(\mathbb{R}^d)$ is the group of linear maps with determinant 1. We will think of Γ as either a group of linear transformations or as a group of matrices.

Our main result is the following.

THEOREM 1.1. *Suppose that $d \geq 2$ and μ is not a point mass. Let G be a closed subgroup of Γ. The corresponding invariant σ-field $\mathcal{I}_G$ consists of sets with probability* 0 *or* 1 *if and only if G is not compact.*

We prove Theorem 1.1 in Section 3 after some preparatory results in Section 2. The proof also uses some consequences of the Jordan canonical form for matrices that are not similar to orthogonal matrices. We establish the relevant results in Section 4.

REMARK 1.1. We note that a closed subgroup G of Γ is compact if and only if there is an invertible matrix h such that $h^{-1}Gh \subseteq \mathrm{O}(\mathbb{R}^d)$. This fact follows from general Lie group theory and is well-known, but we have found an explicit statement with a self-contained accompanying proof to be somewhat elusive. For the sake of completeness, we note the following simple bare hands proof based on Weyl's "unitarian trick". Let η be the normalized Haar measure on G. Define a real inner product $\langle \cdot, \cdot \rangle_\eta$ on $\mathbb{R}^d$ (the elements of which are thought of as column vectors) by $\langle x, y \rangle_\eta := \int_G (gx)^\top (gy)\, \eta(dg)$, where $u^\top$ denotes the transpose of the vector u. It is clear that $\langle gx, gy \rangle_\eta = \langle x, y \rangle_\eta$ for any $g \in G$ and $x, y \in \mathbb{R}^d$. There is a positive definite symmetric matrix S such that $\langle x, y \rangle_\eta = x^\top S y$ (see Exercise 14 in Section 7.2 of [**3**]). Let $h = S^{-\frac{1}{2}}$ be the inverse of the usual positive definite symmetric square root of S (see Theorem 7.2.6 of [**3**]). Then,

$$\begin{aligned}(h^{-1}ghx)^\top (h^{-1}ghy) &= x^\top h g^\top h^{-1} h^{-1} g h y = x^\top h g^\top S g h y \\ &= \langle ghx, ghy \rangle_\eta = \langle hx, hy \rangle_\eta = x^\top h S h y = x^\top y.\end{aligned}$$

Thus, $h^{-1}gh$ preserves the usual Euclidean inner product on $\mathbb{R}^d$ and is an orthogonal matrix, as required.

Suppose that G is compact and h is such that $h^{-1}Gh$ consists of orthogonal matrices. Let U be the closed unit ball in $\mathbb{R}^d$ for the usual Euclidean metric. Then, $g(hU) = (hU)$ for all $g \in G$. Conversely, suppose that G is a closed subgroup of Γ such that $gK \subseteq K$ for all $g \in G$, where K is a compact set with 0 in its interior. It follows that the ℓ^2 operator norms of the elements of G are bounded, and hence G is compact.

We conclude this introduction with some comments about the motivations that led us to consider the question we address in this paper.

A first motivation comes from the forthcoming paper [**2**] on "deterministic Poisson thinning" that we heard about in a lecture by Omer Angel during the 2009 Seminar on Stochastic Processes held at Stanford University.

Let M be the space of non-negative integer valued Radon measures on $\mathbb{R}^d$ for which all atoms are of mass 1 (that is, M is the space of possible realizations of a *simple point process* on $\mathbb{R}^d$). Note that M may be viewed as a subset of Σ. Equip M with the vague topology. It is shown in [**2**] that for $0 < \alpha < \beta$ there is a Borel measurable map $\Theta : M \to M$ such that $\Theta(m) \leq m$ for all $m \in M$ and if Π

is a homogeneous Poisson process with intensity β, then $\Theta(\Pi)$ is a homogeneous Poisson process with intensity α. Moreover, if G is the group of affine Euclidean isometries of $\mathbb{R}^d$, then $\Theta \circ T_g = T_g \circ \Theta$ for all $g \in G$.

It is natural to ask if this equivariance property can hold for some larger group G of affine Lebesgue measure preserving maps. Suppose that this is possible. Take $\mathbb{P}$ to be the distribution of the homogeneous Poisson process with intensity β. Write $\mathbb{P}^x$, $x \in \mathbb{R}^d$, for the associated family of Palm distributions. That is, $\mathbb{P}^x$ is, heuristically speaking, the distribution of a pick from $\mathbb{P}$ conditioned to have an atom of mass 1 at x. In this Poisson case, $\mathbb{P}^x$ is, of course, just the distribution of the random measure obtained by taking a pick from $\mathbb{P}$ and adding an extra atom at x. It follows from the equivariance of Θ under G that if we let H be the subgroup of G that fixes 0, then the map $\gamma : M \to \{0,1\}$ given by $\gamma(m) = (\Theta(m))(\{0\})$ has the property $\gamma \circ T_h = \gamma$, $\mathbb{P}^0$-a.s. for all $h \in H$, and $\mathbb{P}^0\{\gamma = 1\} = \frac{\alpha}{\beta}$. Consequently, if we define $\epsilon : M \to \{0,1\}$ by

$$\epsilon(m) = \begin{cases} \gamma(m + \delta_0), & \text{if } m(\{0\}) = 0, \\ \gamma(m), & \text{otherwise,} \end{cases}$$

where δ_0 is the unit point mass at 0, then $\epsilon \circ T_h = \epsilon$, $\mathbb{P}$-a.s. for all $h \in H$, and $\mathbb{P}\{\epsilon = 1\} = \frac{\alpha}{\beta}$.

However, Theorem 1.1 says that this is impossible if H strictly contains the group $\mathrm{O}(\mathbb{R}^d)$ of linear Euclidean isometries.

A second motivation comes from an analogy with a result in [**4**]. Suppose now that $\mathbb{P}$ is the distribution of a simple point process on $\mathbb{R}^d$. If $\mathbb{P}$ is invariant for all the transformations T_g, $g \in G$, where G is the group of all bijections that preserve Lebesgue measure, then it follows from de Finetti's theorem that $\mathbb{P}$ is of the form $\int \mathbb{Q}^\alpha \, q(d\alpha)$, where $\mathbb{Q}^\alpha$ is the distribution of the homogeneous Poisson process on $\mathbb{R}^d$ with intensity α and the mixing measure q is a probability measure on the nonnegative real numbers. This result may be thought of as a continuum analogue of the special case of de Finetti's theorem which says that an exchangeable sequence of $\{0,1\}$ valued random variables is a mixture of independent, identically distributed, Bernoulli sequences. A counterexample is presented in [**4**] (see also [**5**]) demonstrating that if G is replaced by the smaller group of affine Lebesgue measure preserving transformations, then such a conclusion is false.

In the same way that this result addresses continuum analogues of de Finetti's theorem for small groups of measure preserving transformations, it is natural to consider whether there are continuum analogues of the Hewitt–Savage zero–one law for such groups.

2. Preparatory results

Without loss of generality, we may suppose from now on that $\Omega = \Sigma$, Π is the "canonical Σ-valued random variable" that maps a point $\omega \in \Omega$ to the measure that assigns mass $\omega(B)$ to $B \in \mathcal{B}(\mathbb{R}^d)$, and $\mathcal{F}$ is the $\mathbb{P}$-completion of $\mathcal{S}$. As usual, when speaking of values of random variables we will not explicitly mention points of Ω. In particular, from now on when we use notation such as $\Pi(B)$, the argument is an element of $\mathcal{B}(\mathbb{R}^d)$ and we are referring to the real-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by $\omega \mapsto \omega(B)$.

Write $\mathcal{N}$ for the sub-σ-field of $\mathcal{F}$ consisting of sets with probability 0 or 1. Given $B \in \mathcal{B}(\mathbb{R}^d)$, set $\mathcal{F}_B := \sigma\{\Pi(C) : C \in \mathcal{C}(\mathbb{R}^d),\ C \subseteq B\} \vee \mathcal{N}$. Note for $g \in G$ that if $\Psi : \Omega \to \mathbb{R}$ is $\mathcal{F}_B$-measurable, then $\Psi \circ T_{g^{-1}}$ is $\mathcal{F}_{gB}$-measurable, and, moreover, if $\Upsilon : \Omega \to \mathbb{R}$ is $\mathcal{F}_{gB}$-measurable, then $\Upsilon = \Psi \circ T_{g^{-1}}$ for some $\mathcal{F}_B$-measurable Ψ. Note also that $\mathcal{F}_{B'} \subseteq \mathcal{F}_{B''}$ when $B' \subseteq B''$.

LEMMA 2.1. *Suppose that $\Phi : \Omega \to \mathbb{R}_+$ is a bounded $\mathcal{I}_G$-measurable function. Then, for $g \in G$ and $B \in \mathcal{B}(\mathbb{R}^d)$,*

$$\mathbb{E}\left[\Phi \,|\, \mathcal{F}_B\right] = \mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right] \circ T_g.$$

Consequently, the distribution of $\mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right]$ does not depend on $g \in G$.

PROOF. By the remarks prior to the the statement of the lemma, $\mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right] \circ T_g$ is $\mathcal{F}_B$-measurable. Moreover, if $\Psi : \Omega \to \mathbb{R}_+$ is any bounded $\mathcal{F}_B$-measurable function, then

$$\begin{aligned}\mathbb{E}\left[\Phi \times \Psi\right] &= \mathbb{E}\left[\left(\Phi \circ T_{g^{-1}}\right) \times \left(\Psi \circ T_{g^{-1}}\right)\right] = \mathbb{E}\left[\Phi \times \left(\Psi \circ T_{g^{-1}}\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right] \times \left(\Psi \circ T_{g^{-1}}\right)\right] = \mathbb{E}\left[\left(\mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right] \circ T_g\right) \times \left(\Psi \circ T_{g^{-1}} \circ T_g\right)\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right] \circ T_g\right) \times \Psi\right],\end{aligned}$$

and so $\mathbb{E}\left[\Phi \,|\, \mathcal{F}_{gB}\right] \circ T_g$ is $\mathbb{E}\left[\Phi \,|\, \mathcal{F}_B\right]$, as claimed. □

Denote by $\mathcal{K}(\mathbb{R}^d)$ the collection of compact subsets of $\mathbb{R}^d$.

LEMMA 2.2. *For any $B \in \mathcal{B}(\mathbb{R}^d)$, the σ-fields $\mathcal{F}_B$ and $\sigma\{\Pi(C) : C \in \mathcal{K}(\mathbb{R}^d),\ C \subseteq B\} \vee \mathcal{N}$ coincide.*

PROOF. Suppose that $C \in \mathcal{C}(\mathbb{R}^d)$. By the inner regularity of Lebesgue measure, there exist compact sets $C_1 \subseteq C_2 \subseteq \ldots \subseteq C$ such that $\lim_{n\to\infty} \Lambda(C_n) = \Lambda(C)$. We have $\Pi(C) = \Pi(C_n) + \Pi(C \setminus C_n)$ almost surely. Also, $\Pi(C \setminus C_n)$ has distribution μ_{ℓ_n}, where $\ell_n = \Lambda(C \setminus C_n)$, and so $\Pi(C \setminus C_n)$ converges to 0 in probability as $n \to \infty$. Hence, there exists a subsequence (n_k) such that $\Pi(C \setminus C_{n_k})$ converges to 0 almost surely as $k \to \infty$, so that $\Pi(C_{n_k})$ converges to $\Pi(C)$ almost surely. The result follows directly from this observation. □

LEMMA 2.3. *Suppose that $A_h \in \mathcal{B}(\mathbb{R}^d)$, $h \in \mathbb{Z}$, is a family of sets with the properties $A_{h'} \subseteq A_{h''}$ for $h' < h''$, $\Lambda\left(\bigcap_{h\in\mathbb{Z}} A_h\right) = 0$, and $\Lambda\left(\mathbb{R}^d \setminus \bigcup_{h\in\mathbb{Z}} A_h\right) = 0$. Then, $\bigcap_{h\in\mathbb{Z}} \mathcal{F}_{A_h} = \mathcal{N}$ and $\bigvee_{h\in\mathbb{Z}} \mathcal{F}_{A_h} = \mathcal{F}$.*

PROOF. Consider the claim regarding $\bigcap_{h\in\mathbb{Z}} \mathcal{F}_{A_h}$. It suffices to show that if Ψ is any bounded, non-negative, $\mathcal{F}$-measurable random variable, then $\mathbb{E}\left[\Psi \,|\, \bigcap_{h\in\mathbb{Z}} \mathcal{F}_{A_h}\right]$ is almost surely constant. By the reverse martingale convergence theorem, the latter random variable is almost surely $\lim_{h\to-\infty} \mathbb{E}\left[\Psi \,|\, \mathcal{F}_{A_h}\right]$.

Set $B = \mathbb{R}^d \setminus \bigcap_{h\in\mathbb{Z}} A_h$. Note for any $C \in \mathcal{C}(\mathbb{R}^d)$ that $\Pi(C) = \Pi(C \cap \bigcap_{h\in\mathbb{Z}} A_h) + \Pi(C \cap B) = \Pi(C \cap B)$ almost surely because $\Lambda(C \cap \bigcap_{h\in\mathbb{Z}} A_h) = 0$, and hence $\mathcal{F} = \mathcal{F}_B$. Thus, by Lemma 2.2, $\mathcal{F} = \sigma\{\Pi(C) : C \in \mathcal{K}(\mathbb{R}^d),\ C \subseteq B\} \vee \mathcal{N}$.

Therefore, given any $\epsilon > 0$ there exist compact subsets $C_1, \ldots, C_n$ of B and a bounded Borel function $F : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\mathbb{E}\left[\left|\Psi - F(\Pi(C_1), \ldots, \Pi(C_n))\right|\right] < \epsilon,$$

and so

$$\mathbb{E}\left[\left|\mathbb{E}\left[\Psi \,|\, \mathcal{F}_{A_h}\right] - \mathbb{E}\left[F(\Pi(C_1), \ldots, \Pi(C_n)) \,|\, \mathcal{F}_{A_h}\right]\right|\right] < \epsilon$$

for all $h \in \mathbb{Z}$.

When h is sufficiently small, the compact sets $C_1, \ldots, C_n$ are all contained in the complement of A_h. In that case, the random variable $F(\Pi(C_1), \ldots, \Pi(C_n))$ is independent of the σ-field $\mathcal{F}_{A_h}$ and hence $\mathbb{E}\left[F(\Pi(C_1),\ldots,\Pi(C_n)) \,|\, \mathcal{F}_{A_h}\right]$ is almost surely constant. Therefore, $\mathbb{E}\left[\Psi \,|\, \bigcap_{h\in\mathbb{Z}} \mathcal{F}_{A_h}\right]$ is within $L^1(\mathbb{P})$ distance ϵ of a constant for all $\epsilon > 0$ and so this random variable is itself almost surely constant, as required.

The claim regarding $\bigvee_{h\in\mathbb{Z}} \mathcal{F}_{A_h}$ can be established similarly, and we leave the proof to the reader. □

3. Proof of Theorem 1.1

Suppose that the group G is compact. By Remark 1.1, there is an invertible matrix h such that $g(hU) = (hU)$ for all $g \in G$, where U is the closed unit ball around 0 in $\mathbb{R}^d$ for the usual Euclidean metric. The random variable $\Pi(hU)$ is $\mathcal{I}_G$-measurable and, by the assumption on μ, has distribution $\mu_{\Lambda(hU)}$ that is not concentrated at a point. Therefore, $\mathcal{I}_G$ contains sets that have probability strictly between 0 and 1.

Conversely, suppose that the closed group G is not compact. Then, by Theorem 1 of [**6**], there is matrix $g \in G$ such that the cyclic group $\{g^h : h \in \mathbb{Z}\}$ does not have a compact closure. We note that this result is non-trivial and is related to the "Auerbach problem" – see also [**8, 7**].

Let $(D_t)_{0<t<\infty}$ be the corresponding increasing family of closed subsets of $\mathbb{R}^d$ guaranteed by Lemma 4.1 below. Set $\mathcal{G}_t = \mathcal{F}_{D_t}$. Because $\Lambda\left(\mathbb{R}^d \setminus \bigcup_{0<t<\infty} D_t\right) = 0$, it follows from Lemma 2.3 that $\mathcal{F} = \bigvee_{0<t<\infty} \mathcal{G}_t$.

Suppose, contrary to the statement of the theorem, that there is a bounded $\mathcal{I}_G$-measurable function $\Phi : \Omega \to \mathbb{R}_+$ that is not almost surely equal to a constant. By the martingale convergence theorem,

$$\Phi = \mathbb{E}\left[\Phi \,\middle|\, \bigvee_{n=1}^{\infty} \mathcal{G}_n\right] = \lim_{n\to\infty} \mathbb{E}\left[\Phi \,|\, \mathcal{G}_n\right], \quad \mathbb{P}\text{-a.s.},$$

where the limit is taken over the positive integers. Consequently, there is a positive integer N such that $\mathbb{E}\left[\Phi \,|\, \mathcal{G}_N\right]$ is not $\mathbb{P}$-almost surely equal to a constant. In particular, the variance of $\mathbb{E}\left[\Phi \,|\, \mathcal{G}_N\right]$ is strictly positive.

Because $\Lambda\left(\bigcap_{0<t<\infty} D_t\right) = 0$, it follows from Lemma 2.3 that $\bigcap_{0<t<\infty} \mathcal{G}_t = \mathcal{N}$. For a positive integer n, set $\mathcal{H}_n = \mathcal{G}_{\frac{N}{n}}$. Note that $\mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \ldots$ and $\bigcap_{n=1}^{\infty} \mathcal{H}_n = \mathcal{N}$. By the reverse martingale convergence theorem,

$$\lim_{n\to\infty} \mathbb{E}\left[\Phi \,\middle|\, \mathcal{H}_n\right] = \mathbb{E}\left[\Phi \,\middle|\, \bigcap_{n=1}^{\infty} \mathcal{H}_n\right] = \mathbb{E}\left[\Phi\right], \quad \mathbb{P}\text{-a.s. and in } L^2(\mathbb{P}).$$

In particular, the variance of $\mathbb{E}\left[\Phi \,|\, \mathcal{H}_n\right]$ converges to 0 as $n \to \infty$.

For a non-negative integer m, set $\mathcal{E}_m = \mathcal{F}_{g^m D_N}$. Thus, $\mathcal{E}_0 = \mathcal{G}_N = \mathcal{H}_1$. It follows from Lemma 2.1 that the distribution of $\mathbb{E}\left[\Phi \,|\, \mathcal{E}_m\right]$ is that of $\mathbb{E}\left[\Phi \,|\, \mathcal{G}_N\right]$ for all m. In particular, the variance of $\mathbb{E}\left[\Phi \,|\, \mathcal{E}_m\right]$ is the same as that of $\mathbb{E}\left[\Phi \,|\, \mathcal{G}_N\right]$ for all m. Because $g^h D_{t''} \subseteq D_{t'}$ for $0 < t' < t'' < \infty$ and h sufficiently large, we have for any given positive integer n that there exists an integer m for which $\mathcal{E}_m \subseteq \mathcal{H}_n$. In that case,

$$\mathbb{E}\left[\mathbb{E}\left[\Phi \,|\, \mathcal{E}_m\right]\right] = \mathbb{E}[\Phi] = \mathbb{E}\left[\mathbb{E}\left[\Phi \,|\, \mathcal{H}_n\right]\right]$$

and, by the conditional Jensen's inequality,

$$\mathbb{E}\left[\mathbb{E}\left[\Phi \,|\, \mathcal{E}_m\right]^2\right] \le \mathbb{E}\left[\mathbb{E}\left[\Phi \,|\, \mathcal{H}_n\right]^2\right]$$

so that the variance of $\mathbb{E}[\Phi \mid \mathcal{E}_m]$ is dominated by the variance of $\mathbb{E}[\Phi \mid \mathcal{H}_n]$.

The former variance does not depend on m and is strictly positive, whereas the latter variance converges to 0 as $n \to \infty$, so we arrive at a contradiction.

4. Consequences of the Jordan canonical form

Let A be a $d \times d$ matrix with entries from the field $\mathbb{C}$ of complex numbers. For convenience, we say that A has *order* d. We recall some facts from linear algebra that may be found, for example, in Ch. 3 of [**3**].

The *geometric multiplicity* of an eigenvalue λ of A is the dimension of the null space of the matrix $\lambda I - A$ (that is, the geometric multiplicity is the maximal number of linearly independent solutions of the equation $Ax = \lambda x$, $x \in \mathbb{C}^d$). The *algebraic multiplicity* of the eigenvalue λ is the multiplicity of λ as a root of the characteristic equation $t \mapsto \det(tI - A)$ (that is, the algebraic multiplicity is the largest positive integer m such that the polynomial $(t-\lambda)^m$ divides the polynomial $\det(tI - A)$).

Suppose that the sum of the geometric multiplicities of the eigenvalues of A is k. Because eigenvalues corresponding to distinct eigenvalues are linearly independent, k is the dimension of the sum of the null spaces of $\lambda I - A$ as λ ranges over the eigenvalues of A.

For a positive integer r and $\zeta \in \mathbb{C}$, let $J_r(\zeta)$ be the $r \times r$ matrix given by

$$J_r(\zeta)_{ij} := \begin{cases} \zeta, & i = j, \\ 1, & j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

That is, every entry of $J_r(\zeta)$ on the diagonal is ζ, every entry on the super-diagonal is 1, and every other entry is 0.

There exists an invertible matrix S with entries from $\mathbb{C}$ such that $J := S^{-1}AS$ is block diagonal with blocks $J_{d_1}(\lambda_1), \ldots, J_{d_k}(\lambda_k)$. The numbers $\lambda_1, \ldots, \lambda_k$ are all eigenvalues of A, with each distinct eigenvalue appearing at least once. The geometric multiplicity of an eigenvalue λ is the number of times that λ appears in the list $\lambda_1, \ldots, \lambda_k$. The algebraic multiplicity of λ is the sum of the orders of the corresponding blocks. The matrix J, which is unique up to a re-ordering of the λ_p, is the *complex Jordan canonical form* of A.

An order $2r$ matrix of the form

$$\begin{pmatrix} J_r(\zeta) & 0 \\ 0 & J_r(\bar{\zeta}) \end{pmatrix},$$

where $\bar{\zeta} = c - id$ is the complex conjugate of $\zeta = c + id$, is similar to a block matrix $C_r(\zeta)$ in which each block has order 2, the diagonal blocks are all of the form

$$\begin{pmatrix} c & d \\ -d & c \end{pmatrix},$$

the super-diagonal blocks are all identity matrices, and the remaining blocks are all 0 matrices.

Suppose now that the entries of A are in $\mathbb{R}$. Define $\lambda_1, \ldots, \lambda_k$ and J as before. If some λ_p is not real, then its complex conjugate $\bar{\lambda}_p$ appears as λ_q for some q with $d_p = d_q$. There is an invertible matrix T with entries from $\mathbb{R}$ such that $K := T^{-1}AT$ is block diagonal with blocks $J_{a_1}(\eta_1), \ldots, J_{a_s}(\eta_s), C_{b_1}(\kappa_1), \ldots, C_{b_t}(\kappa_t)$. The numbers $\eta_1, \ldots, \eta_s$ are the real eigenvalues in the list $\lambda_1, \ldots, \lambda_k$ while the numbers

$\kappa_1, \dots \kappa_t$ come from picking one member of each complex conjugate pair of non-real eigenvalues in the list. If $\eta_m = \lambda_\ell$, then $a_m = d_\ell$, and if $\kappa_n \in \{\lambda_p, \lambda_q\}$ with $d_p = d_q$, then $b_n = d_p = d_q$. The matrix K is the *real Jordan canonical form* of A.

Suppose now that $A \in \Gamma = (+1) \times \mathrm{Sl}(\mathbb{R}^d) \sqcup (-1) \times \mathrm{Sl}(\mathbb{R}^d)$, so that $\det A = \lambda_1^{d_1} \cdots \lambda_k^{d_k} \in \{\pm 1\}$. The matrix A is similar to an orthogonal matrix if and only if $k = d$ (equivalently, $d_1 = \cdots = d_k = 1$) and $|\lambda_1| = \cdots = |\lambda_k| = 1$. Also, A is similar to an orthogonal matrix if and only if the group of matrices $\{A^h : h \in \mathbb{Z}\}$ is bounded. Hence, if $\{A^h : h \in \mathbb{Z}\}$ is not bounded, then A is not similar to an orthogonal matrix and either $|\lambda_\ell| < 1$ for some ℓ or $|\lambda_1| = \cdots = |\lambda_k| = 1$ and $d_\ell \geq 2$ for some ℓ. It follows that one or more of the blocks in the real Jordan canonical form of A must have one of the following forms:

a) $J_a(\eta)$ with $a \geq 2$ and $\eta \in \{\pm 1\}$,
b) $C_b(\kappa)$ with $b \geq 2$ and $\kappa \in \{z \in \mathbb{C} \setminus \mathbb{R} : |z| = 1\}$,
c) $J_a(\eta)$ with $a \geq 1$ and $\eta \in \{x \in \mathbb{R} : |x| < 1\}$,
d) $C_b(\kappa)$ with $b \geq 1$ and $\kappa \in \{z \in \mathbb{C} \setminus \mathbb{R} : |z| < 1\}$.

Consider case (a). Note that $J_a(\eta) = \eta I + N_a$, where $N_a := J_a(0)$ is the order a matrix with 1 at every position on the super-diagonal and 0 elsewhere. Write $e_1, \dots, e_a$ for the standard basis of column vectors of $\mathbb{R}^d$; that is, e_i has 1 in the i^{th} coordinate and 0 elsewhere. Observe that

$$N_a^j e_i = \begin{cases} e_{i-j}, & \text{if } j < i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} J_a(\eta)^h (x_1 e_1 + \cdots + x_a e_a) &= \sum_{i=1}^a x_i \sum_{j=0}^h \binom{h}{j} \eta^{h-j} N_a^j e_i \\ &= \sum_{i=1}^a x_i \sum_{j=0}^{i-1} \binom{h}{j} \eta^{h-j} e_{i-j} = \sum_{i=1}^a x_i \sum_{\ell=1}^{i} \binom{h}{i-\ell} \eta^{h-i+\ell} e_\ell \\ &= \eta^h \sum_{\ell=1}^a \left[\sum_{i=\ell}^a x_i \frac{h(h-1)\cdots(h-(i-\ell))}{(i-\ell)!} \eta^{-(i-\ell)} \right] e_\ell. \end{aligned}$$

Write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the usual Euclidean inner product and norm on $\mathbb{R}^a$, and set $V_\rho := \{x \in \mathbb{R}^a : |\langle x, e_a \rangle| \leq \rho \|x\|\}$, for $0 < \rho < 1$. Note that the sets V_ρ have the properties

- $V_{\rho'} \subset V_{\rho''}$ for $0 < \rho' < \rho'' < 1$,
- $\bigcup_{0<\rho<1} V_\rho = \mathbb{R}^a \setminus \{t e_a : t \neq 0\}$,
- $\bigcap_{0<\rho<1} V_\rho = \{x \in \mathbb{R}^a : \langle x, e_a \rangle = 0\}$.

In particular, the sets $\mathbb{R}^a \setminus \bigcup_{0<\rho<1} V_\rho$ and $\bigcap_{0<\rho<1} V_\rho$ are both Lebesgue null. It is not difficult to see from the above that

$$J_a(\eta)^h V_{\rho''} \subseteq V_{\rho'}, \quad 0 < \rho' < \rho'' < 1, \quad h \text{ sufficiently large.}$$

A similar argument shows that the conclusion of the previous paragraph hold in case (c), provided $a \geq 2$. If $a = 1$ in case (c), then the sets $B_\epsilon := \{x \in \mathbb{R} : |x| \leq \epsilon\}$, $0 < \epsilon < \infty$, have the properties

- $B_{\epsilon'} \subset B_{\epsilon''}$ for $0 < \epsilon' < \epsilon'' < \infty$,
- $\bigcup_{0<\epsilon<\infty} B_\epsilon = \mathbb{R}$,

- $\bigcap_{0<\epsilon<\infty} B_\epsilon = \{0\}$.

In particular, the sets $\mathbb{R} \setminus \bigcup_{0<\epsilon<\infty} B_\epsilon$ and $\bigcap_{0<\epsilon<\infty} B_\epsilon$ are both Lebesgue null. It is clear that

$$J_1(\eta)^h B_{\epsilon''} \subseteq B_{\epsilon'}, \quad 0 < \epsilon' < \epsilon'' < \infty, \quad h \text{ sufficiently large.}$$

Analogous constructions for the cases (b) and (d) and some straightforward further argument complete the proof of the following result.

LEMMA 4.1. *Suppose that the matrix $A \in \Gamma$ is such that the cyclic group $\{A^h : h \in \mathbb{Z}\}$ does not have a compact closure. Then, there exists a collection $(D_t)_{0<t<\infty}$ of closed subsets of $\mathbb{R}^d$ with the following properties:*

- $D_{t'} \subset D_{t''}$ *for* $0 < t' < t'' < \infty$,
- $\Lambda\left(\mathbb{R}^d \setminus \bigcup_{0<t<\infty} D_t\right) = 0$,
- $\Lambda\left(\bigcap_{0<t<\infty} D_t\right) = 0$,
- $A^h D_{t''} \subseteq D_{t'}$ *for* $0 < t' < t'' < \infty$ *and h sufficiently large.*

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E-mail address: `evans@stat.Berkeley.EDU`

DEPARTMENT OF STATISTICS #3860, UNIVERSITY OF CALIFORNIA AT BERKELEY, 367 EVANS HALL, BERKELEY, CA 94720-3860, U.S.A